Stern-Gerlach Interaction in Fermion Beams

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Abstract. The Stern-Gerlach interaction, between a moving charged particle endowed with a magnetic moment and a radio-frequency e.m. field, is studied by means of a semi-classical approach. Theoretical results are presented, and a possible experimental check of this theory is discussed.

In the example of a charged particle with magnetic moment which travels inside a time varying electromagnetic field, the expression of the Stern-Gerlach force in the laboratory frame has been deduced [1,2] by means of some quite complicated calculations. In fact we had to start from the particle rest frame (x', y', z', t') where such a force assumes the usual form

$$\vec{f}'_{SG} = \nabla'(\vec{\mu}^* \cdot \vec{B}') = \frac{\partial}{\partial x'}(\vec{\mu}^* \cdot \vec{B}')\hat{x} + \frac{\partial}{\partial y'}(\vec{\mu}^* \cdot \vec{B}')\hat{y} + \frac{\partial}{\partial z'}(\vec{\mu}^* \cdot \vec{B}')\hat{z}, \tag{1}$$

where $\vec{\mu}^* = g\frac{e}{2m}\vec{S}$ is the magnetic moment. The partial derivatives and the fields $\vec{E}, \vec{B}, \vec{E}', \vec{B}'$ are Lorentz boosted along the z-axis via the following transformations:

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial y'} = \frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial z'} = \gamma\left(\frac{\partial}{\partial z} + \frac{\beta}{c}\frac{\partial}{\partial t}\right), \tag{2}$$

$$\vec{E}' = \gamma(\vec{E} + c\vec{\beta} \times \vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta} \cdot \vec{E}), \tag{3}$$

and

$$\vec{B}' = \gamma\left(\vec{B} - \frac{\vec{\beta}}{c} \times \vec{E}\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta} \cdot \vec{B}). \tag{4}$$

Moreover, bearing in mind that the force transforms as

$$\vec{f}_\perp = \frac{1}{\gamma}\vec{f}'_\perp, \qquad \vec{f}_\parallel = \vec{f}'_\parallel, \quad \text{and} \quad f_z = f'_z, \tag{5}$$

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we obtain

$$\vec{f}_{SG} = \frac{1}{\gamma}\frac{\partial}{\partial x}(\vec{\mu}^* \cdot \vec{B}')\hat{x} + \frac{1}{\gamma}\frac{\partial}{\partial y}(\vec{\mu}^* \cdot \vec{B}')\hat{y} + \frac{\partial}{\partial z'}(\vec{\mu}^* \cdot \vec{B}')\hat{z} = f_x\hat{x} + f_y\hat{y} + f_z\hat{z} \tag{6}$$

with

$$f_x = \mu_x^*\left(\frac{\partial B_x}{\partial x} + \frac{\beta}{c}\frac{\partial E_y}{\partial x}\right) + \mu_y^*\left(\frac{\partial B_y}{\partial x} - \frac{\beta}{c}\frac{\partial E_x}{\partial x}\right) + \frac{1}{\gamma}\mu_z^*\frac{\partial B_z}{\partial x}, \tag{7}$$

$$f_y = \mu_x^*\left(\frac{\partial B_x}{\partial y} + \frac{\beta}{c}\frac{\partial E_y}{\partial y}\right) + \mu_y^*\left(\frac{\partial B_y}{\partial y} - \frac{\beta}{c}\frac{\partial E_x}{\partial y}\right) + \frac{1}{\gamma}\mu_z^*\frac{\partial B_z}{\partial y}, \tag{8}$$

$$f_z = \mu_x^* C_{zx} + \mu_y^* C_{zy} + \mu_z^* C_{zz} \tag{9}$$

where

$$C_{zx} = \gamma^2\left[\left(\frac{\partial B_x}{\partial z} + \frac{\beta}{c}\frac{\partial B_x}{\partial t}\right) + \frac{\beta}{c}\left(\frac{\partial E_y}{\partial z} + \frac{\beta}{c}\frac{\partial E_y}{\partial t}\right)\right], \tag{10}$$

$$C_{zy} = \gamma^2\left[\left(\frac{\partial B_y}{\partial z} + \frac{\beta}{c}\frac{\partial B_y}{\partial t}\right) - \frac{\beta}{c}\left(\frac{\partial E_x}{\partial z} + \frac{\beta}{c}\frac{\partial E_x}{\partial t}\right)\right], \tag{11}$$

$$C_{zz} = \gamma\left(\frac{\partial B_z}{\partial z} + \frac{\beta}{c}\frac{\partial B_z}{\partial t}\right). \tag{12}$$

Let us consider a rectangular resonator, whose sides a, b, and c are respectively parallel to the x, y, and z-axes, and which is excited in the TE_{011} mode. If the spin orientation is 50% parallel and 50% antiparallel to $\vec{B}_{\mathrm{ring}} \parallel \hat{y}$, as commonly assumed for describing an unpolarized fermion beam, and if we choose $x = \frac{a}{2}$ and $y = \frac{b}{2}$ as beam coordinates, such a cavity is characterized [3] by the following parameters

$$K_c = \frac{\pi}{b}, \quad \omega = c\sqrt{\left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{d}\right)^2}, \quad \beta_{\mathrm{ph}} = \frac{d}{\pi}\sqrt{\left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{d}\right)^2} \tag{13}$$

with $v_{\mathrm{ph}} = \beta_{\mathrm{ph}} c$ = wave's phase velocity, and field components

$$\vec{B} = \begin{cases} 0 \\ -B_0\frac{b}{d}\cos\left(\frac{\pi z}{d}\right)\cos\omega t \\ 0 \end{cases}, \quad \vec{E} = \begin{cases} -\omega\, B_0\frac{b}{\pi}\sin\left(\frac{\pi z}{d}\right)\sin\omega t \\ 0 \\ 0 \end{cases}. \tag{14}$$

Therefore the most important force-component is

$$f_z = \mu^*\gamma^2 B_0 b\left\{\frac{1}{\pi}\left[\left(\frac{\pi}{d}\right)^2 + \left(\frac{\beta\omega}{c}\right)^2\right]\sin\left(\frac{\pi z}{d}\right)\cos\omega t + \frac{2}{d}\left(\frac{\beta\omega}{c}\right)\cos\left(\frac{\pi z}{d}\right)\sin\omega t\right\}, \tag{15}$$

whose integration over the cavity length d gives the following expression of the energy gained, or lost, by a fermion which crosses the cavity:

$$\Delta U = \int_0^d f_z dz = \mu^*\gamma^2 B_0\frac{b}{d}\frac{\left(\frac{\pi}{d}\right)^2 + \left(\frac{\beta\omega}{c}\right)^2 - \left(\frac{\omega}{c}\right)^2}{\left(\frac{\pi}{d}\right)^2 - \left(\frac{\omega}{\beta c}\right)^2}\left[1 + \cos\left(\frac{\omega d}{\beta c}\right)\right], \tag{16}$$

having carried out the substitution $\omega t = \frac{\omega z}{\beta c}$. The stationary wave conditions, pertaining to the TE_{011} mode, imply that $d = \frac{1}{2}\beta_{\mathrm{ph}}\lambda$; hence Eq. (16) becomes

$$\Delta U = \mu^* B_0 \frac{b}{d}\left[\beta^2\gamma^2 \frac{\beta_{\rm ph}^2 - 1}{\beta_{\rm ph}^2 - \beta^2} + \frac{\beta_{\rm ph}^2 \beta^2}{\beta_{\rm ph}^2 - \beta^2}\right]\left(1 + \cos\tfrac{\beta_{\rm ph}}{\beta}\pi\right), \tag{17}$$

or in the ultrarelativistic limit ($\gamma \gg 1$ and $\beta \simeq 1$)

$$\Delta U \simeq \mu^* B_0 \frac{b}{d}\gamma^2 (1 + \cos\beta_{\rm ph}\pi) = 2\,\mu^* B_0 \frac{b}{d}\gamma^2 \quad (\beta_{\rm ph} = \text{even integer}). \tag{18}$$

This energy exchange has to be compared to the one caused by the electric field, whose evaluation [1,2] yields

$$\Delta U_E = \int_0^d eE_x\,dx = \left[e\omega B_0 \frac{bd}{\pi^2}\frac{\beta^2}{\beta_{\rm ph}^2 - \beta^2}\sin\frac{\beta_{\rm ph}}{\beta}\pi\right] x' \simeq \left[\frac{bd}{2\pi}\frac{\beta_{\rm ph}}{\beta_{\rm ph}^2 - 1}\frac{e\omega B_0}{\gamma^2}\right] x', \tag{19}$$

where x' is the trajectory slope, for $\beta_{\rm ph}$ equal to an **integer** and for ultrarelativistic particles.

As far as a spin-splitter [1,2] is concerned, we recall that spin up particles receive (or loose) that amount of energy given by Eq. (18) at each rf cavity crossing. Simultaneously, spin down particles behave exactly in the opposite way, i.e. they loose (or gain) the same amount of energy turn after turn. The most important issue is that the transferred energies add up coherently, i.e. the final energy separation after N_{SS} revolutions is

$$\Delta U_{\uparrow\downarrow} = \sum\{\Delta U_\uparrow - (-\Delta U_\downarrow)\} = 4\frac{b}{d}N_{SS}\,\mu^* B_0\,\gamma^2 \simeq 4\,N_{SS}\,\mu^* B_0\,\gamma^2. \tag{20}$$

Summing up the energy contributions (19) from the electric field gives

$$(\Delta U_E)_{tot} = \sum \Delta U_E = \kappa \sum x' = 0, \tag{21}$$

since the sign of x' changes continuously due to the incoherence of betatron oscillations.

Recalling that the spin-splitter principle requires a repetitive crossing of N_{cav} cavities, and that after each revolution the particle experiences a deviation of its momentum spread

$$\zeta = \frac{\delta p}{p} = \frac{1}{\beta^2}\frac{\delta E}{E} \simeq \frac{N_{cav}\Delta U}{E} \simeq \frac{2\sqrt{3}}{3}\,N_{cav}\frac{B_0}{B_\infty}\gamma, \tag{22}$$

having made use of Eq. (18) with $\beta_{\rm ph} = 2$ and $B_\infty = \frac{mc^2}{\mu^*} \simeq 10^{16}\,T$ for (anti)protons. From Eq. (22) we find that the number of turns and the time Δt needed for attaining a momentum separation equal to $2(\Delta p/p)$ are respectively

$$N_{SS} = \frac{(\Delta p/p)}{\zeta} = \frac{1}{2\,N_{cav}\gamma}\frac{B_\infty}{B_0}\left(\frac{\Delta p}{p}\right), \quad \text{and} \quad \Delta t = N_{SS}\tau_{\rm rev}. \tag{23}$$

TABLE 1. RHIC and HERA parameters

	RHIC	HERA
E(GeV)	250	820
γ	266.5	874.2
$\tau_{\rm rev}(\mu s)$	12.8	21.1
$\frac{\Delta p}{p}$	4.1×10^{-3}	5×10^{-5}
N_{SS}	6.67×10^{9}	2.48×10^{7}
Δt	$8.52 \times 10^{4}\ s \simeq 23.7\ h$	523 s
μ^*	$1.41 \times 10^{-26}\ JT^{-1}$	

TABLE 2. MIT-Bates parameters

$\tau_{\rm rev}$	634 nsec	b/d	$\simeq 1$
$\omega_{\rm rev}/2\pi$	1.576 MHz	B_0	$\simeq 0.1$ T
$N_{electrons}$	$3.6 \times 10^{8} \cdot 225 = 8.1 \times 10^{10}$	$\omega_{\rm rf}/\omega_{\rm rev}$	1820
γ	978.5	μ^*	$9.27 \times 10^{-24}\ JT^{-1}$

Table 1 shows estimates for RHIC [4] and HERA [5] with

$$B_0 \simeq 0.1\ T, \quad \text{and} \quad N_{cav} = 200.$$

Let us now evaluate a possible experimental test [6,7] of a polarimeter in the MIT-Bates [8] ring (see Table 2), loaded with 500 MeV polarized electrons. For a single rectangular cavity, with peak magnetic field B_0, and for a bunch train made up of N particles with polarization P, the average power transferred is the ratio between the total energy transfer of Eq. (18) and the revolution period, namely

$$W \simeq 2NP\frac{\mu^* B_0}{\tau_{\rm rev}}\,\frac{b}{d}\,\gamma^2. \tag{24}$$

If we adopt a two coupled cavity scheme as a parametric amplifier, similar to the one proposed for a different [9] application, we may obtain a huge amplification of the small signal generated by the Stern-Gerlach interaction. Two cavities, tuned at the same frequency and coupled either in a symmetric or antisymmetric mode, can act as a parametric converter [10,11] provided that the frequency separation between the two modes is equal to the revolution frequency of the beam. With an initially empty level, the power transferred to this level is

$$W_2 = \frac{\omega_{\rm rf}}{\omega_{\rm rev}}\,W \simeq 2\,N\,P\,\frac{\mu^* B_0}{\tau_{\rm rf}}\,\frac{b}{d}\,\gamma^2 \simeq 431\,P \text{ watt}, \tag{25}$$

where the cavity's period is $\tau_{\rm rf} = 1/f_{\rm rf}$ and $f_{\rm rf} \sim 3$ GHz.

Comparing the energy exchanges (18) and (19) for $x' \simeq 1$ mrad, $\beta_{\rm ph} = 2$ and $\lambda = 10$ cm in the MIT-Bates ring, we obtain:

$$r = \frac{\Delta U_E}{\Delta U} = \frac{x'}{8}\,\frac{\beta_{\rm ph}^3}{\beta_{\rm ph}^2 - 1}\,\frac{\lambda e c}{\mu^*}\,\frac{1}{\gamma^4} = 1.72 \times 10^{-4}, \tag{26}$$

i.e. the spurious signal, depending upon the electric interaction is negligible with respect to the measurable signal generated by the magnetic interaction. However, numerical simulations with spin-tracking [12] and cavity-designing [13] codes should be made by considering a real machine with a system of either rectangular or cylindrical cavities.

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