Hadron beam-beam diffusion in 2.5-D

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Abstract

The standard analysis of modulational diffusion for general nonlinearities is qualitatively summarized, and compared to the particular case of a beam-beam simulation with two kicks per turn, plus tune modulation. A simulation with realistic Tevatron parameters shows amplitude growth over long timescales of order 10^4 synchrotron periods.

The simulated amplitude growth is qualitatively similar to the predictions of modulational diffusion, showing large discrete steps in the evolution speed as the tune distance from the nearest 2-D weak coupling resonance is varied. However, the simulation shows a fundamental difference in that the observed amplitude growth is approximately exponential in time, and not approximately root time as predicted in the standard analysis. Possible reasons for this and other discrepancies are briefly discussed.

1 MODULATIONAL DIFFUSION

Models of 1-D nonlinear motion in a hadron collider have timescales of hundreds of turns. When externally driven tune modulation is added, "thick layer" chaos may ensue [1, 2, 3]. If chaos is present in such a 1.5-D model, the typical timescale is of order a hundred modulation periods. Synchrotron oscillations generate pseudo-external tune modulation, coupling through non-zero chromaticity. However, even a hundred synchrotron periods amount to only a few seconds, far shorter than the empirically observed timescales of the order of hours [4, 5, 6, 7].

Arnol'd "thin layer" diffusion occurs in 2-D models, with very long timescales of hundreds of millions of turns, but it is generally conceded to be too weak to be a significant practical concern [8, 9, 10]. Of the classical catalog of diffusion mechanisms, only modulational "thick layer" diffusion in 2.5-D is strong enough and long enough in time scale to be a serious candidate for a successful description of the evolution of hadron collider bunch distributions. Unfortunately, even modulational diffusion has serious deficiencies in explaining the basic features of very simple 2.5-D beam-beam simulations.

A general "standard analysis" of modulational diffusion is well reported in the literature, complete with analytical and quantitative results [9, 10, 11]. These results are not discussed in detail here. Instead, the general characteristics of modulational diffusion are described, and the requirements for its existence are summarized.

1.1 Thick layer chaos in 1.5-D

First, consider nonlinear motion in only the horizontal dimension, in the vicinity of a one-dimensional resonance, and under the influence of tune modulation, described by

$$Q_x = Q_{x0} + q\sin(2\pi Q_M t) \tag{1}$$

Here t is time measured in machine turns, Q_{x0} is the unperturbed horizontal base tune, q is the tune modulation depth, and Q_M is the tune modulation tune. Depending on the location in tune modulation parameter space (q, Q_M), the motion falls roughly into one of four dynamical phases – "chaos", "strong sidebands", "amplitude modulation", or "phase modulation" [12, 13, 14].

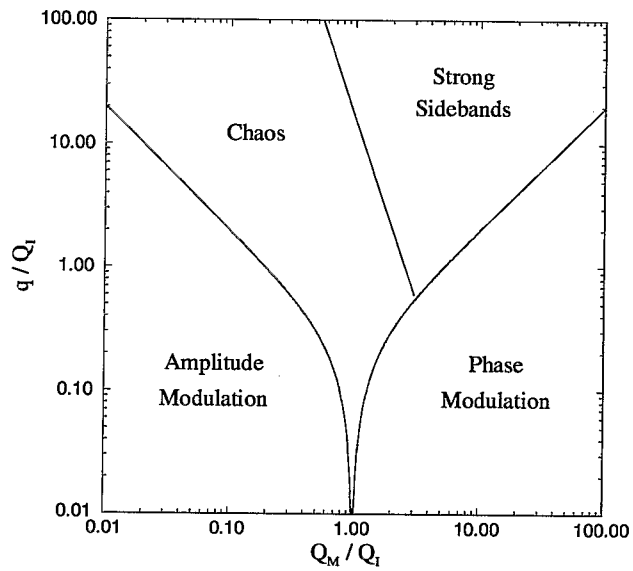

Figure 1: Dynamical phases in the tune modulation parameter plane (q, Q_M), for a resonance of order $N = 5$.

The boundaries between these dynamical phases are smoothly drawn in Figure 1. In this figure both q and Q_M scale with the parameter Q_I, the "island tune" corresponding to small oscillations around the stable fixed point at the center of a resonance island. The straight line limits of the smooth boundaries in Figure 1 are consistent with Hamiltonian approximations which are valid when q and/or Q_M are incommensurate with Q_I. From left to right these straight line segments are

$$\left(\frac{q}{Q_I}\right)\left(\frac{Q_M}{Q_I}\right) = \frac{1}{N} \tag{2}$$

$$\left(\frac{q}{Q_I}\right)^{1/4}\left(\frac{Q_M}{Q_I}\right)^{3/4} = \frac{4}{(N\pi)^{1/4}} \tag{3}$$

$$\left(\frac{q}{Q_I}\right)\left(\frac{Q_M}{Q_I}\right)^{-1} = \frac{1}{N} \tag{4}$$

where N is the order of the 1-D resonance. The curved portions of the boundaries are drawn with some artistic li-

cense. To put it simply, dynamical stability is vulnerable to tune modulation tunes near to the island tune [15].

Simulations show that, although the true boundaries have a detailed complex structure, nonetheless the smooth boundaries are approximately correct [13]. Most important of all, simulations confirm the universal dependence of the boundaries on only three configuration parameters: (q/Q_I), (Q_M/Q_I), and N. The details of the nonlinearity – its distribution and its source (beam-beam or magnetic) – do not matter except through the island tune, Q_I.

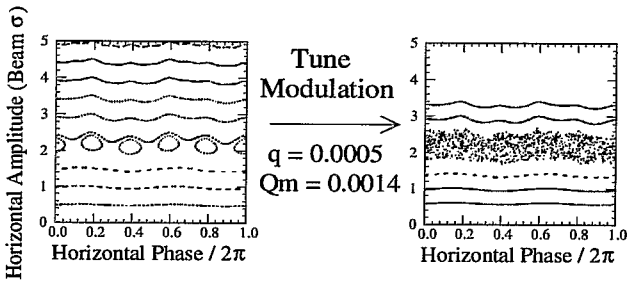

Figure 2: The transition to "thick layer" chaos in horizontal phase space when tune modulation is turned on in the presence of a $Q_x = 3/5$ resonance. These results come from a simulation of the Tevatron with two beam-beam collisions of strength $\xi = 0.0005$, offset by $\Delta x = 0.1\sigma$.

A thick layer of *bounded* chaos is formed when the tune modulation parameters (q,Q_M) lie in the dynamical phase labeled "chaos" in Figure 1. This is illustrated in Figure 2. It typically takes a particle of order a hundred modulation periods to move all the way across the chaotic layer.

The classical general model of modulational diffusion requires a thick chaotic layer in (say) horizontal phase space, to act as a noise source for the vertical motion. It also requires that the vertical motion is coupled to the horizontal through a weak 2-D coupling resonance. The standard analysis then proceeds by reducing the vertical motion to a random walk in which a test particle tends to diffuse to large vertical amplitudes.

1.2 *The modulational diffusion coefficient, D*

Chirikov [11] and Lichtenberg and Lieberman [10] discuss a standard Hamiltonian for modulational diffusion

$$\begin{aligned} H &= \frac{1}{2}J_x^2 - k\cos[\theta_x + \lambda \sin \Omega t] \\ &\quad + \frac{1}{2}J_y^2 - \epsilon \cos[\theta_x - \theta_y] \end{aligned} \tag{5}$$

where (J, θ) are the action-angle variables in each plane. Analogies are easily drawn between the parameters of this standard model and those appropriate to accelerators, if t in Equation 5 is interpreted as turn number. For example, $k \rightleftharpoons (2\pi Q_I)^2$ (resonance strength), $\lambda \rightleftharpoons (q/Q_M)$ (modulation strength), and $\Omega \rightleftharpoons (2\pi Q_M)$ (modulation frequency).

The standard analysis predicts that motion scales as the square root of time, and defines a local diffusion coefficient

$$D \equiv \frac{\langle \Delta J_y^2 \rangle}{2T} \tag{6}$$

where ΔJ_y is the vertical action excursion and T is the elapsed time. The ensemble averaging should be performed over a time T short compared to the vertical diffusion time (so that $\Delta J_y/J_y$ is small) but long compared to the timescales of horizontal motion across the thick chaotic band (of order a hundred synchrotron periods).

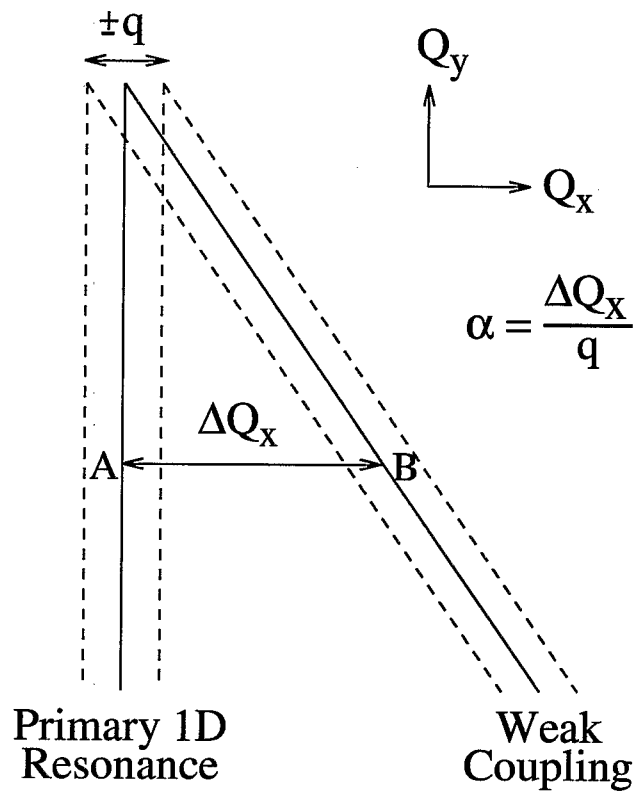

Figure 3: Modulational diffusion resonance structure. The resonance separation is parameterized by α, which is the horizontal tune distance ΔQ_x scaled by the primary resonance half width – the tune modulation depth q.

It turns out that the diffusion coefficient D depends strongly on α, the proximity of the weak 2-D resonance. The test particle has an instantaneous tune at "A" in Figure 3, deep in a thick layer of horizontal chaos around a primary horizontal resonance which is $\pm q$ wide. A single weak coupling resonance lies a distance ΔQ_x away, a distance which is measured in units of the primary resonance half width

$$\alpha = \frac{\Delta Q_x}{q} \tag{7}$$

After a *tour de force* derivation including a number of assumptions and approximations, the standard analysis predicts that $D(\alpha)$ drops by about 16 orders of magnitude as α is increased from 0 to about 7, in a dramatic series of sudden descents at odd integer values of α.

The literature shows that there is good (although not perfect) agreement between the analytical prediction for $D(\alpha)$ and simulation results obtained by numerical integration of the differential equations of motion [10, 11]. One free parameter is adjusted to optimize the agreement. The predicted "plateaux and cliffs" are clearly visible.

Unfortunately this is not a realistic scenario for a hadron collider – not even a small fraction of a bunch population would (deliberately) be placed in the middle of a strong chaotic resonance! More relevant to long term collider behavior is the scenario of a test particle at location "B" in Figure 3 – it is never possible for all of the bunch population to avoid all weak 2-D resonances. In this scenario the parameter α would measure the proximity of the primary resonance, and *not* the weak coupling resonance. Also it is often not realistic to assume that there are only two simple resonances in the tune vicinity of a test particle.

Do the striking plateaux and cliffs in $D(\alpha)$ also occur when discrete nonlinear maps representing turn-by-turn collider motion replace continuous differential equations?

To answer this question, this paper turns to a simple realistic model of beam-beam collisions in the Tevatron. However, the question is valid for either magnetic or beam-beam nonlinearities. Note that the preceding discussion of modulational diffusion has been completely general, without specifying the particular source of the nonlinearity.

2 BEAM-BEAM NONLINEARITIES

The strong-weak beam-beam interaction is a good candidate for a quantitative analysis of 2.5-D amplitude growth, because it is mathematically well behaved [16]. In particular:

- The beam-beam kick goes like $1/R$ at large amplitudes, so the tune shifts and the resonance strengths all go to zero.
- At all amplitudes the quantities of interest – the detuning, resonance strengths, island widths, and the island tunes, et cetera – are simply linear in the small quantity ξ, the beam-beam tune shift parameter.
- Analytical expressions can be written for the quantities of interest, at all amplitudes.

None of these statements are true for magnetic nonlinearities. All of them are readily confirmed by tracking.

As an example of the analytical tractability of the beam-beam interaction, consider the island tune on a primary 1-D resonance due to a single head on beam-beam interaction. It is given by

$$Q_I(J) = N\xi\left(|V_N(J)|\,U''(J)\right)^{1/2} \tag{8}$$

where the resonance order N is even, J is the action at the resonance, $V_N(J)$ is the "resonance width function", and $U''(J)$ is the second derivative of the function $U(J)$ with respect to J. These two function are given, in turn, by integrating the following expressions

$$V_N'(J) = -(-1)^{N/2}\frac{4}{J}\exp\left(\frac{-J}{2}\right)I_{N/2}\left(\frac{J}{2}\right) \tag{9}$$

$$U'(J) = \frac{2}{J}\left[1-\exp\left(\frac{J}{2}\right)I_0\left(\frac{J}{2}\right)\right] \tag{10}$$

where $I_{N/2}$ and I_0 are modified Bessel functions.

The Hamiltonian which approximately describes the motion near the $Q = p/N$ beam-beam resonance is

$$\frac{H}{2\pi} = (Q_0 - \frac{p}{N})J + \xi U(J) + \xi V_N(J)\cos N\theta \tag{11}$$

where Q_0 is the base tune [16]. It is straightforward to extend this formalism to include include odd order resonances driven by (small) closed orbit offsets. It is also possible to extend the analytical description from 1-D to 2-D, although not without some pain.

3 A SIMPLE REALISTIC BEAM-BEAM SIMULATION

There were two strong beam-beam interactions on every turn during the 1992 run of the Fermilab Tevatron – one at the CDF experiment and one at the D0 experiment – each with a strength of $\xi \approx 0.005$ [17]. Unwanted collisions at other locations were avoided by the use of electrostatic separators. Typical operating parameters are listed in Table 1. The working point caused the tune footprint to lie between 7th and 5th order resonances, straddling 12th order sum resonances, as shown in Figure 4.

Quantity	Label	Value
Base tunes	Q_{x0}, Q_{y0}	20.586, 20.575
Chromaticities	χ_x, χ_y	3.0
RMS momentum spread	σ_p/p	0.0003
Synchrotron tune	Q_M	0.00078
Beam-beam parameter	ξ	0.005

Table 1: Typical Tevatron operational parameters at 900 GeV in the 1992 collider run.

The weak-strong tracking program EVOL was used for all simulations [18]. The beam-beam interactions assumed round Gaussian beams of transverse size σ, so that the horizontal beam-beam kick was

$$\Delta x' = \frac{-4\pi\,\xi}{\beta^*}\frac{2(x+\Delta x)}{R^2}\left[1-e^{-R^2/2}\right] \tag{12}$$

Here x and x' are the horizontal displacement and angle, β^* is the beta function at the interaction point (the same in

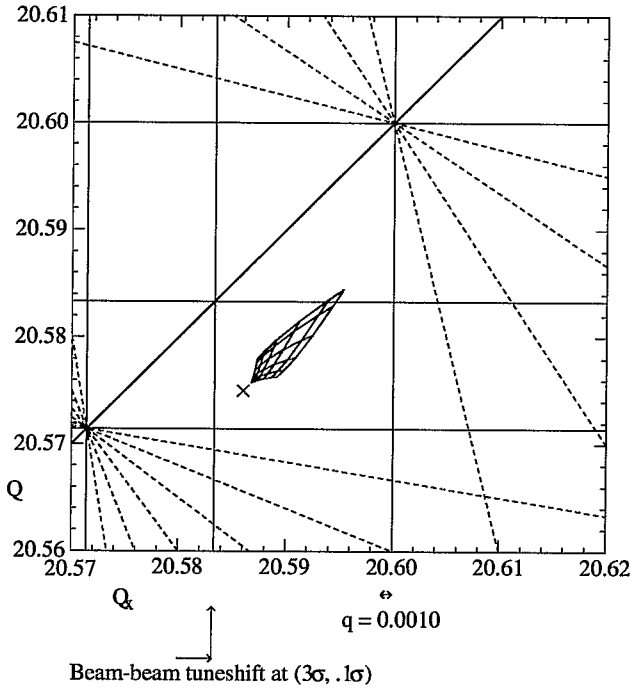

Figure 4: The tune plane for 1992 Tevatron parameters, showing the 5th, 7th, and 12th order resonances. The nominal working point is indicated by a cross, and the beam-beam footprint for $\xi = 0.005$ at two collisions is shown. Footprint contours of constant amplitude range from 0.1σ to 5.1σ in 1σ increments.

both planes), and R is the distance from the center of the opposing beam scaled to the beam size σ

$$R^2 \equiv \left(\frac{(x+\Delta x)}{\sigma}\right)^2 + \left(\frac{y}{\sigma}\right)^2 \qquad (13)$$

A similar kick was applied in the vertical plane.

A small horizontal closed orbit offset of $\Delta x = 0.1\sigma$ was included at the collision points, so the 5th order resonance $Q_x = 20.6$ was driven directly. Although the actual value of this offset was not routinely measured during the 1992 run, collision offsets of this magnitude were considered quite possible.

A tune modulation depth of $q = 0.001$ was used, present only in the horizontal plane, corresponding to a horizontal chromaticity of about 3 units combining with an off-momentum amplitude of $\Delta p/p \approx 3 \times 10^{-4}$, a realistic value for the Tevatron.

4 SIMULATION RESULTS

The maximum vertical amplitude was recorded for test particles launched with initial amplitudes $(a_x, a_y) = (3.0, 0.1)\sigma$, for tracking times ranging from 10 to 10^4 synchrotron periods [13]. This is as long as a few minutes in the Tevatron.

4.1 *Tune plan scans*

First, a tune plane scan was performed over a mesh on the tune plane diagram of Figure 4, giving the results shown in Figure 5. The strongest vertical amplitude growth is seen

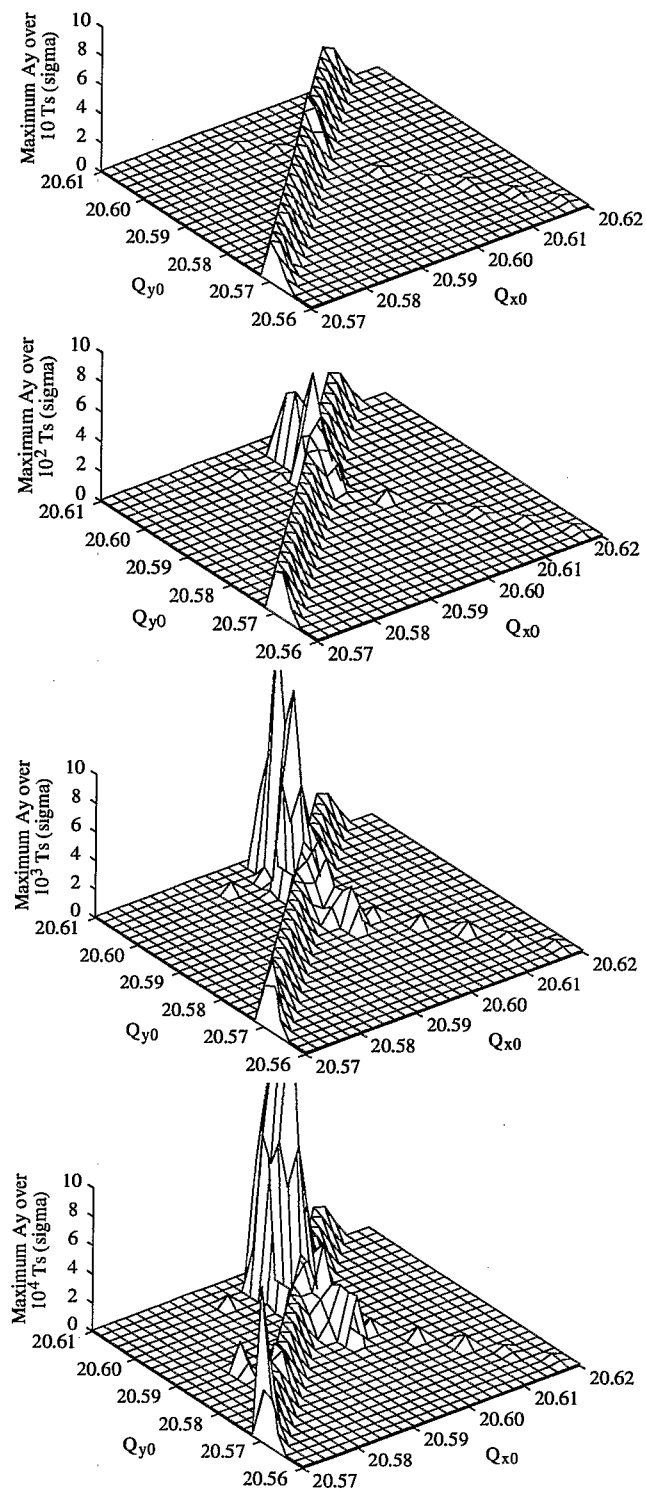

Figure 5: Maximum vertical amplitudes of single particles with initial amplitudes $(a_x, a_y) = (3.0, 0.1)\sigma$, tracked for 10, 10^2, 10^3 and 10^4 synchrotron periods.

near the intersection of the $Q_y = Q_x$ and $Q_x = 20.6$ resonances. Relatively modest growth is seen elsewhere, for example in the vicinity of the $3Q_x+2Q_y = 103$ resonance. The range of timescales observed is qualitatively consistent with that predicted for modulational diffusion.

Vertical amplitude growth is almost completely absent with the tune modulation turned off ($q = 0$), except in the vicinity of the $Q_y = Q_x$ linear coupling resonance. This is conclusive evidence that tune modulation drives amplitude growth on these timescales.

Careful observation reveals that the resonances visible in Figure 5 are slightly displaced from their nominal locations. This is due to the "detuning" of the instantaneous tunes (Q_x, Q_y) from the base tune values (Q_{x0}, Q_{y0}) used as axes in the figure. The size of the tune shift for a test particle with $(a_x, a_y) \approx (3.0, 0.1)\sigma$ can be estimated from the tune footprint shown in Figure 4.

Tune variation with amplitude is not included in the standard modulational diffusion Hamiltonian of Equation 5. Also, the variation of resonance strength with both amplitudes, which is explicitly present in the beam-beam case, is not present in the modulational diffusion model.

4.2 *Primary resonance scan*

A second scan was performed by setting $Q_{x0} = 20.597$, and decreasing the vertical base tune Q_{y0} to gradually increase α, the distance to the nearest coupling resonance. This value of Q_{x0} places a test particle in the center of a thick layer of chaos surrounding the $Q_x = 20.6$ resonance, a necessary condition for modulational diffusion predictions to apply. Note that the nearest coupling resonance $4Q_x + Q_y = 103$ is not directly visible in Figure 5.

Figure 6 shows the vertical amplitude evolution for 3 quite different values of α over times as long as 6×10^3 synchrotron periods. Tracking was stopped when the vertical amplitude reached 1 σ, in order to avoid complicating the interpretation of the results with amplitude dependent effects. For each value of α the evolution is plotted, side by side, on both log-log and log-linear scales. The vertical amplitude clearly evolves like an exponential of time ($a_y \sim \exp \gamma t$), and not like root time ($a_y \sim t^{1/2}$) as predicted by the standard modulational diffusion model, or by standard diffusion phenomenology.

Figure 7 shows exponential growth rate data from simulations over a range of α values. In each case the exponential growth rate γ was extracted by fitting the the raw data to a curve

$$a_y(t) = 0.1\sigma\, e^{\gamma t/T_s} \tag{14}$$

where T_s is the synchrotron period. Two distinct plateaux are visible in Figure 7, separated by sudden drops at values $\alpha = 2$ and $\alpha = 3$, each of about 2 orders of magnitude. In contrast, the standard model predicts cliffs only at odd integer values of α.

The growth rate γ has natural units of inverse synchrotron periods, since the natural time unit is one modulation period. It is therefore inappropriate to include data

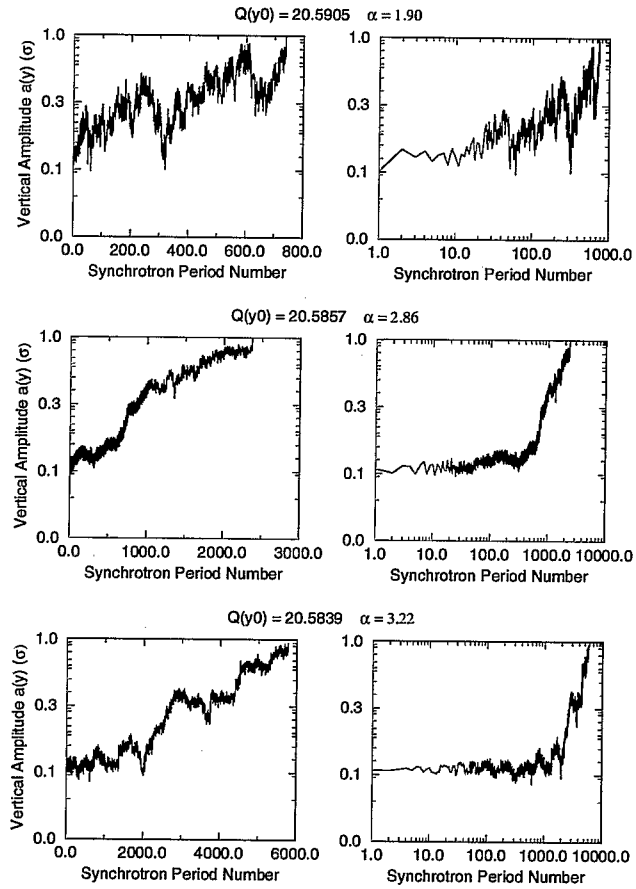

Figure 6: Exponential vertical amplitude growth for particles launched within a horizontal chaotic band. The base tune is $Q_{x0} = 20.597$ in all cases, so that the horizontal motion sits in the middle of the $Q_x = 20.6$ resonance. Note the different timescales.

sets with fast growth rates $\gamma > 1$, limiting α to values larger than about 2. The maximum value of $\alpha \approx 4.5$ is given by the availability of cpu time, and human patience.

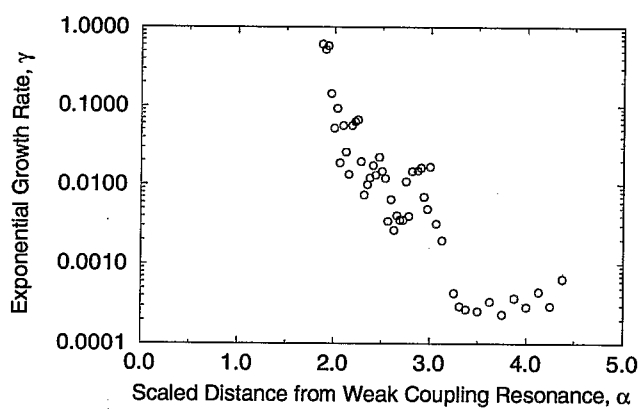

Figure 7: Exponential vertical amplitude growth rate γ plotted versus the scaled distance α from the $4Q_x + Q_y = 106$ weak coupling resonance.

5 CONCLUSIONS

The analytical theory of modulational diffusion has been investigated by comparing its predictions with the results of a simple simulation of beam-beam behavior in the Tevatron. The simulation confirms that particles subject to "thick layer" horizontal chaos on the $Q_x = 20.6$ resonance experience a vertical amplitude growth over timescales of tens of thousands of synchrotron periods, or several minutes.

No particles sit in a chaotic layer around the $Q_x = 20.6$ resonance under normal operating conditions in the Tevatron. The strongest primary resonance which in practice might provide a noise source for modulational diffusion is $12Q_y = 247$. Typical timescales observed in the Tevatron are on the order of hours.

The rate of simulated vertical amplitude growth depends on the proximity of a nearby coupling resonance, and shows a structure of "plateaux and cliffs" similar to modulational diffusion predictions. However, the growth is exponential in time, and not root-time as predicted.

Resonance strengths in the standard modulational diffusion model are not amplitude dependent. It has been suggested [19] that the exponential growth seen in the beam-beam simulation may be due to the variation of primary or coupling resonance strength as the vertical amplitude increases from 0.1σ to 1.0σ. This seems unlikely, in the restricted range of horizontal and vertical amplitudes which were tracked, but it remains a viable possibility.

The 2.5-D beam-beam interaction (with tune modulation) is analytically quite tractable, in addition to being a simple problem of practical concern. It may be possible to extend the standard analytical treatment of modulational diffusion to this case, and in doing so to include the important features of detuning, and resonance strength variation with amplitude.

An even simpler 2.5-D collider model, that of 3 octupoles plus 1 decapole, has already been used for other tune modulation studies [13]. The octupoles are arranged to provide linear detuning with action without driving any resonances, while the decapole drives a resonance with no detuning. This model is more artificial than the beam-beam model, but it is also more tractable analytically (at modest amplitudes). It is also extremely fast in simulation.

The arena of 2.5-D diffusion is ripe for further study, in an attempt to reconcile analytical predictions with simple simulations. The ideal simple simulation would bear some relevance to contemporary – and future – hadron collider performance.

6 REFERENCES

[1] F. Izrailev, S. Misnev, G. Tumaikin, Preprint 77-43, Inst. Nucl. Phys. Novosibirsk (1977).

[2] B. Chirikov, Phys. Rep. 52: 265 (1979).

[3] E. Courant, BNL-28186, Upton, BNL (1980).

[4] T. Chen et. al. Phys. Rev. Lett. 68: 33 (1992).

[5] W. Fischer, "An Experimental Study on the Long-Term Stability of Particle Motion in Hadron Storage Rings", Dissertation, U. Hamburg (1995).

[6] O. Brüning and F. Willeke, Phys. Rev. Lett. 76: 3719 (1996).

[7] W. Fischer et al, Phys. Rev. E, 55: 3507, (1997).

[8] B. Chirikov, J. Ford, F. Izrailev, Preprint 81-70, Inst. Nucl. Phys. Novosibirsk (1981).

[9] F. Vivaldi, "Weak Instabilities in Many Dimensional Hamiltonian Systems", Rev. Mod. Phys. 56: 737 (1984).

[10] A. Lichtenberg and M. Lieberman, "Regular and Chaotic Dynamics", 2nd Edition, Springer, New York, (1991).

[11] B. Chirikov, M. Lieberman, F. Vivaldi and D.L. Shepelyanski, "A Theory of Modulational Diffusion", Physica 14D: 298 (1985).

[12] S. Peggs, "Hamiltonian Theory of the E778 Nonlinear Dynamics Experiment", Proc. 2nd Advanced ICFA Beam Dynamics Workshop, Lugano, CERN 88-04 and SSC-175, (1988).

[13] T. Satogata, "Nonlinear Resonance Islands and Modulational Effects in a Proton Synchrotron", Ph.D. Thesis, Northwestern University (1993).

[14] T. Satogata et. al. "Driven Response of a Trapped Particle Beam", Phys. Rev. Lett. 68: 1838 (1992).

[15] T. Sen and J. Ellison, "Diffusion due to Beam-Beam Interaction and Fluctuating Fields in Hadron Colliders ", Phys. Rev. Lett. 77: 1051 (1996).

[16] S. Peggs and R. Talman, "Nonlinear Problems in Accelerator Physics", Ann. Rev. Nucl. Part. Sci. 36: 287 (1986).

[17] S. Holmes, "Achieving High Luminosity in the Fermilab Tevatron", PAC 91, San Francisco, 2986 (1991).

[18] S. Peggs, "Hadron Collider Behavior in the Nonlinear Numerical Model EVOL", Particle Accelerators 17: 11 (1985).

[19] O. Brüning, private communication, (1992).