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TUNE SPLITTING IN THE PRESENCE OF LINEAR COUPLING*

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I. INTRODUCTION

The presence of random skew quadrupole field errors will couple the x and y motions. The x and y motions are then each given by the sum of 2 normal modes with the tunes ν_1 and ν_2, which may differ appreciably from ν_x and ν_y, the unperturbed tunes. This is often called tune splitting since $|\nu_1 - \nu_2|$ is usually larger than $|\nu_x - \nu_y|$. This tune splitting may be large in proton accelerators using superconducting magnets, because of the relatively large random skew quadrupole field errors that are expected in these magnets. This effect is also increased by the required insertions in proton colliders which generate large β–functions in the insertion region.

This tune splitting has been studied in the RHIC accelerator (the Relativistic Heavy Ion Collider proposed at Brookhaven National Laboratory). For RHIC, a tune splitting as large as 0.2 was found in one worse case. A correction system has been developed[1] for correcting this large tune splitting which uses two families of skew quadrupole correctors. It has been found[1] that this correction system corrects most of the large tune splitting, but a residual tune splitting remains that is still appreciable.

RHIC has to operate within a box in tune space whose width is $\Delta\nu = 33 \times 10^{-3}$ in order to avoid all resonances which are tenth order or less. It appears desirable to correct the tune splitting to a level which is much smaller than 33×10^{-3}. The residual tune splitting that remained after correction with the 2 family tune splitting correction system was found to be about 18×10^{-3}.

The residual tune splitting appears to be due to higher order effects of the random a_1 multipole. Loosely speaking, one may say that the residual tune splitting is associated with the nearby linear sum resonance, $\nu_x + \nu_y =$ integer. A skew quadrupole correction system has been developed that appears able to correct a large part of the residual tune splitting. This consists of skew quadrupole correctors near the high–β quadrupoles in the insertions, which are excited so as to generate the harmonics that are near $\nu_x + \nu_y$. An analytic result for the ν shift, that includes higher order terms in a_1, indicates that the important harmonics are the harmonics near $\nu_x + \nu_y$.

II. CALCULATION AND CORRECTION OF THE NORMAL MODE TUNE

The normal mode tunes ν_1 and ν_2 can be computed in the following way. The effect of the a_1 multipole in each magnet is represented by putting point a_1 multipoles at the entrance and exit of each magnet, with half the strength of the a_1 multipole. The one turn transfer matrix around the ring can be computed by multiplying the transfer matrices of all the magnets and all the point a_1 multipoles in the ring. The 4×4 one turn transfer matrix can be written as

$$T = \begin{bmatrix} M & n \\ m & N \end{bmatrix} \tag{2.1}$$

where M, N, m, n are each 2×2 matrices. The tunes ν_1 and ν_2 can then be computed from the following equations[2]

$$\begin{aligned} \cos\mu_1 + \cos\mu_2 &= \frac{1}{2}\, Tr\,(M+N) \\ \cos\mu_1 - \cos\mu_2 &= \left\{ \left[\frac{1}{2} Tr\,(M-N)\right]^2 + |m + \overline{n}| \right\}^{\frac{1}{2}} \\ \overline{n} &= \begin{bmatrix} n_{22} & -n_{12} \\ -n_{21} & n_{11} \end{bmatrix}, \mu_1 = 2\pi\nu_1, \mu_2 = 2\pi\nu_2 \,. \end{aligned} \tag{2.2}$$

Treating the random a_1 as a perturbation one can find an approximate result which is valid close to the resonance line $\nu_x - \nu_y = p$, p being some integer.

$$\begin{aligned} \nu_1 &= \overline{\nu}_x \pm \left[(\nu_x - \nu_y - p)^2/4 + |\Delta\nu\,(\overline{\nu}_x, \overline{\nu}_y)|^2 \right]^{\frac{1}{2}} \\ \nu_2 &= \overline{\nu}_y \pm \left[(\nu_x - \nu_y - p)^2/4 + |\Delta\nu\,(\overline{\nu}_x, \overline{\nu}_y)|^2 \right]^{\frac{1}{2}} \\ \Delta\nu\,(\nu_x, \nu_y) &= (1/4\pi\rho) \int ds\, (\beta_x\beta_y)^{\frac{1}{2}}\, a_1 \exp\left[i\left(-\nu_x\theta_x + \nu_y\theta_y\right)\right] \\ \overline{\nu}_x &= (\nu_x + \nu_y + p)/2, \overline{\nu}_y = (\nu_x + \nu_y - p)/2, \\ \theta_x &= \psi_x/\nu_x, \theta_y = \psi_y/\nu_y \end{aligned} \tag{2.3}$$

$\Delta\nu$ may be regarded as the width of nearby coupling resonance. ν_1 is the ν value that goes to ν_x when $a_1 \to 0$, and similarly for ν_2 and ν_y. For the $\pm$ sign, the top sign is used when $\nu_x > \nu_y + p$ and the bottom sign when $\nu_x < \nu_y + p$. ρ is the radius of curvature in the main dipoles and a_1 is defined by $B_x = B_0 a_1 x$ where B_0 is the main dipole field.

Equations (2.3) suggest that if the a_1 correctors are set so as to make $\Delta\nu = 0$, then the tune splitting is minimized. This is not a practical way to set the a_1 correctors as the $\Delta\nu$ and the a_1 errors are not known for a real accelerator. Usually the a_1 correctors are set so as to minimize the tune splitting which can be measured. The results from this way of setting the a_1 correctors is discussed in Section III. It is instructive to see what happens when the a_1 correctors are set to make $\Delta\nu = 0$. These results are shown in Table 1 for 10 distributions of

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a_1 errors and for a RHIC lattice having $\beta^* = 2$ at all 6 crossing points.

The results in Table 1 were found using the RHIC lattice with the expected[3] random quadrupole errors, a_1 and b_1. The tune of the normal modes, ν_1 and ν_2, are computed using the exact result, Eq. (2.2). The tune shift due to the random b_1 has been corrected using the QF and QD quadrupoles in the regular cells. A correction system has been developed[1] for correcting the tune splitting which uses two families of skew quadrupoles as correctors. Results are given for ν_1, ν_2 before and after the application of the tune splitting correction, and are listed as the uncorrected and corrected cases. The tune splitting correction was set so as to make $\Delta\nu = 0$, where $\Delta\nu$ is the $\nu_x = \nu_y$ coupling resonance width given in Eqs. (2.3).

The unperturbed ν_x, ν_y for RHIC for Table 1 is $\nu_x = 28.826$, $\nu_y = 28.821$. Only the fractional part of the ν–values are listed in Table 1. The largest tune splittings found in Table 1 are $|\nu_1 - \nu_2|/10^{-3}, = 228$ uncorrected and 23 corrected. One sees that the tune splitting remaining after correction is considerably larger than the 5×10^{-3} predicted by Eq. (2.3). The remaining tune splitting after correction is called the residual tune splitting in this paper. It appears to be due to higher order terms in a_1 which are omitted in Eq. (2.3). An extension of Eq. (2.3) to include higher order terms in a_1 has been found, and is given by the two solutions for $\nu_{x,s}$ and $\nu_{y,s}$, $\nu_{x,s} = \nu_{y,s} + p$, of the equations

$$\left(\nu_{x,s}^2 - \tilde{\nu}_x^2\right)\left(\nu_{y,s}^2 - \tilde{\nu}_y^2\right) = 4\nu_x\nu_y|\Delta\nu\left(\nu_{x,s}, \nu_{y,s}\right)|^2$$

$$\tilde{\nu}_x^2 = \nu_x^2 + \Delta_x, \tilde{\nu}_y^2 = \nu_y^2 + \Delta_y$$

$$\Delta_x = 4\nu_x\nu_y \sum_{n\neq -p} |c_n|^2 / \left[(n+p)\left(n+\nu_x+\nu_y\right)\right]$$

$$\Delta_y = 4\nu_x\nu_y \sum_{n\neq p} |b_n|^2 / \left[(n-p)\left(n+\nu_x+\nu_y\right)\right]$$

$$b_n = (1/4\pi\rho)\int ds a_1 \left(\beta_x\beta_y\right)^{\frac{1}{2}} \exp\left[i\left(-\left(n+\nu_y\right)\theta_x + \nu_y\theta_y\right)\right]$$

$$c_n = (1/4\pi\rho)\int ds a_1 \left(\beta_x\beta_y\right)^{\frac{1}{2}} \exp\left[i\left(\nu_x\theta_x - \left(n+\nu_x\right)\theta_y\right)\right] . \tag{2.4}$$

For $p = 0$, the $\nu_x - \nu_y = 0$ resonance, Eqs. (2.4) can be solved to give

$$\nu_1^2 = \left(\tilde{\nu}_x^2 + \tilde{\nu}_y^2\right)/2 \pm \left[\left(\tilde{\nu}_x^2 - \tilde{\nu}_y^2\right)^2/4 + 4\nu_x\nu_y|\Delta\nu\left(\nu_1,\nu_1\right)|^2\right]^{\frac{1}{2}}$$

$$\nu_2^2 = \left(\tilde{\nu}_x^2 + \tilde{\nu}_y^2\right)/2 \mp \left[\left(\tilde{\nu}_x^2 - \tilde{\nu}_y^2\right)^2/4 + 4\nu_x\nu_y|\Delta\nu\left(\nu_2,\nu_2\right)|^2\right]^{\frac{1}{2}} \tag{2.5}$$

For $\Delta\nu\left(\nu_1,\nu_2\right)$, $\Delta\nu\left(\nu_x,\nu_y\right)$ is evaluated for $\nu_x = \nu_y = \nu_1$, where ν_1 is given by Eq. (2.3), and similarly for $\Delta\nu\left(\nu_2,\nu_2\right)$. Eq. (2.5) reduces to Eq. (2.3) when higher order terms in a_1 are dropped. The above equations show that when $\Delta\nu\left(\overline{\nu},\overline{\nu}\right) = 0$ for $\overline{\nu} = \left(\nu_x+\nu_y\right)/2$ and $\nu_x = \nu_y$, there is a residual tune splitting given by $\nu_1 - \nu_2 = \left(\Delta_x - \Delta_y\right)/2\nu_x$, and $\left(\nu_1+\nu_2\right)/2 = \overline{\nu} + \Delta_x/4\nu_x + \Delta_y/4\nu_y$. Thus after a tune splitting correction that makes $\Delta\nu \simeq 0$, the harmonics of a_1 near $\nu_x + \nu_y$ contribute most to the residual $|\nu_1 - \nu_2|$. This result is useful in designing a correction system for the residual tune splitting (see Section III). The derivation of the above result will be given in a future paper.

Table 1: Results for the correction of the tune splitting for a $\beta^* = 2$ RHIC lattice using a 2 family tune splitting correction system set to make $\Delta\nu = 0$.

	—Uncorrected—			—Corrected—		
Error Field Dist.	ν_1	ν_2	$\lvert\nu_1-\nu_2\rvert$ $/10^{-3}$	ν_1	ν_2	$\lvert\nu_1-\nu_2\rvert$ $/10^{-3}$
1	0.796	0.854	59	0.828	0.823	6
2	0.707	0.935	228	0.838	0.819	19
3	0.869	0.783	86	0.825	0.829	4
4	0.772	0.883	111	0.831	0.823	7
5	0.779	0.872	93	0.836	0.820	16
6	0.848	0.805	43	0.832	0.821	11
7	0.840	0.847	7	0.852	0.834	18
8	0.742	0.895	153	0.838	0.818	20
9	0.785	0.866	81	0.828	0.823	6
10	0.749	0.891	142	0.823	0.827	5

III. CORRECTION OF THE RESIDUAL $|\nu_1 - \nu_2|$

A first step in reducing the residual $|\nu_1 - \nu_2|$, is to set the a_1 correctors of the two family tune splitting correction system to minimize the residual $|\nu_1 - \nu_2|$, instead of setting them to make $\Delta\nu = 0$ as indicated by Eqs. (2.3). The results of doing this are given in Table 2. In Table 2, the results for the residual $|\nu_1 - \nu_2|$ listed in Table 1 using the $\Delta\nu = 0$ correction procedure are compared with the results found using the minimizing $|\nu_1 - \nu_2|$ procedure for the same 10 distributions of random a_1 and b_1 errors and for the $\beta^* = 2$ lattice for RHIC. Table 2 shows that the minimizing $|\nu_1 - \nu_2|$ procedure considerably reduces the residual $|\nu_1 - \nu_2|$ for some field error distributions. A possible explanation is that for some error distributions, the setting of the correctors to make $\Delta\nu = 0$ excites the harmonics in a_1 that are near $\nu_x + \nu_y =$ integer, and it is these harmonics that are believed to be responsible for the residual $|\nu_1 - \nu_2|$.

One may note that after correction, not only is there a considerable residual $|\nu_1 - \nu_2|$ for some field error distributions, but also the average tune $\nu_{av} = \left(\nu_1+\nu_2\right)/2$ may be considerably off from $\left(\nu_x+\nu_y\right)/2$. It was found that the location of ν_{av} may be corrected using the normal tune correction system which in RHIC consists of the QF and QD quadrupoles in the regular cells of the arcs which are corrected in two families. This is indicated by Eqs. (2.3). However the normal tune correction system will not change the residual $|\nu_1 - \nu_2|$ by much.

Table 2: Residual tune splitting resulting from two different methods of tune splitting correction using the 2–family tune splitting correction system.

Error Field Distribution	$\lvert\nu_1-\nu_2\rvert/10^{-3}$ $\Delta\nu=0$ Correction	$\lvert\nu_1-\nu_2\rvert/10^{-3}$ minimize $\lvert\nu_1-\nu_2\rvert$ Correction
	$\beta^*=2$	
1	6	5
2	19	18
3	4	3
4	7	7
5	16	8
6	11	11
7	18	7
8	20	16
9	6	6
10	5	4

To further reduce the residual tune splitting requires additional a_1 correctors so that one can control not only the harmonics in a_1 that drive the difference resonance, $\nu_x-\nu_y$, but also the harmonics in a_1 that are near $\nu_x+\nu_y$, which for RHIC are principally the 58 and 57 harmonics in a_1. The most effective place to put these correctors is near the quadrupoles in the insertions that have high β–functions. In RHIC which has 6 beam crossing points, there are 12 such high β locations. The two family tune splitting correction system[1] has a_1 correctors at 6 of the high β locations. The proposed enlargement of the a_1 correctors is to have a_1 correctors at all 12 of the high β locations, and all 12 a_1 correctors are separately powered.

A computer study was done to evaluate the performance of the enlarged a_1 correction system in reducing the residual $|\nu_1-\nu_2|$. The results are given in Table 3 for one particular method of using the a_1 correctors which is described below. One may note that the unperturbed tune of the accelerator in this study has a tune splitting of $|\nu_x-\nu_y|=5\times10^{-3}$, and the results listed in Table 3 for the residual $|\nu_1-\nu_2|$ using the enlarged a_1 correction are good.

The method used to reduce the residual $|\nu_1-\nu_2|$ is the following: the 6 a_1 correctors that make up the 2–family tune splitting correction system are corrected in the usual way as described in Ref. 1. This provides two variables which are primarily used to correct the effects of the difference resonance $\nu_x=\nu_y$. In addition the 12 a_1 correctors near the high β quadrupoles are excited according to the equation

$$a_1 = A\cos(\psi_x+\psi_y)+B\sin(\psi_x+\psi_y),$$

where ψ_x,ψ_y are the betatron phases at the correctors and A and B are two so far unknown variables. This provides 2 more variables used primarily to correct the $\nu_x+\nu_y$. Using the above hook up of the a_1 correctors, the 4 available variables are adjusted to minimize $|\nu_1-\nu_2|$. This gives the results in Table 3.

The hook up of the a_1 correctors described above does not use the full flexibility of the enlarged a_1 correction system, but it appears to do a fairly good job of reducing the residual $|\nu_1-\nu_2|$.

There appears to be some advantages to having some a_1 correctors in the arcs of the accelerator. The a_1 correctors in the arcs cannot usually entirely correct the difference resonance as they tend to have about the same value of $\psi_x-\psi_y$ if they are put near the quadrupoles in the arcs. However if the phase is defined so that $\psi_x-\psi_y\simeq0$ at the quadrupoles, then the arc a_1 correctors can be used to correct the real part of $\Delta\nu$, the width of the difference resonance $\nu_x=\nu_y$, which is often a large part of $\Delta\nu$. The strength required of the a_1 correctors in the insertions is then reduced, and there is some improvement in the performance of the correction system. In RHIC, it is proposed to have a_1 correctors near most of the defocusing quadrupoles in the arcs which are connected in series in one family, which would be used to correct the real part of $\Delta\nu$, the width of the difference resonance. For the results given in Table 3, the arc a_1 correctors were also used.

Table 3: The reduction in the residual tune splitting obtained using the enlarged a_1 correction system, compared with the results obtained with the 2–family correction system, minimizing $|\nu_1-\nu_2|$.

Error Field Distribution	$\lvert\nu_1-\nu_2\rvert/10^{-3}$ 2 Family Correction System	$\lvert\nu_1-\nu_2\rvert/10^{-3}$ enlarged a_1 Correction System
	$\beta^*=2$	
1	5	5
2	16	6
3	3	3
4	5	5
5	5	5
6	9	3
7	14	4
8	10	2
9	6	3
10	1	3

The random skew quadrupole errors also produce an appreciable change[4] in the β– functions of the normal modes, β_1 β_2. The change in β_1, β_2 can be quite large for some distributions of random a_1 errors, as much as 100% has been found for RHIC. The above described enlarged a_1 corrector system has been found to be able to also correct most of the change in β_1, β_2. It was also found[5] that correction of the tune splitting also corrected the change in the β–functions to a considerable extent.

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