Andrew Meyer - DOB 12/28/76 Graduate Application - Dept. of Physics

Estimation of Tune and Chromaticity from Schottky Spectra

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Efficient methods for estimation of tune and chromaticity from Schottky spectra are presented. We also present an approximation of Schottky spectra as a sum of Gaussians.

Schottky Spectra

In the limit of linear longitudinal dynamics, the power spectrum of transverse Schottky signals is

$$(1) \qquad F(\square) = \sum_{n=\square} \left[\begin{array}{l} A \sum_{k=\square} I_k\left(\square_t^2\left(\square \square\left(\square_\square + \square_\square\right)\right)^2\right) \exp\left(\square \square_t^2\left(\square \square\left(\square_\square + \square_\square\right)\right)^2\right) \square\left(\square \square\left(\square_\square + n\square_r + k\square_s\right)\right) + \\ B \sum_{k=\square} I_k\left(\square_t^2 \square^2\right) \exp\left(\square \square_t^2 \square^2\right) \square\left(\square \square\left(n\square_r + k\square_s\right)\right) + \\ A \sum_{k=\square} I_k\left(\square_t^2\left(\square + \left(\square_\square + \square_\square\right)\right)^2\right) \exp\left(\square \square_t^2\left(\square + \left(\square_\square + \square_\square\right)\right)^2\right) \square\left(\square \square\left(\square\square_\square + n\square_r + k\square_s\right)\right) \end{array} \right.$$

where $\square_t$ is the RMS bunch length, $\square_r$ is the angular revolution frequency, and $\square_s$ is the angular synchrotron frequency. The betatron tune $\square_\square$ and chromaticity $\square$ are contained in the expressions $\square_\square \equiv \square_\square \square_r$ and $\square_\square \equiv \square \square_\square \square_r / \square$. (In this paper, after Siemann, $\square \equiv E \square \square_\square / \square E \square_\square$, in contrast to the RHIC convention $\square \equiv P \square \square_\square / \square_\square$.) A and B are normalization constants related to the sensitivity of the Schottky cavity. [Siemann]

Under realistic conditions, $\square_s << \square_r$, so that qualitatively the spectrum consists of a pattern of three groups of closely-spaced delta functions corresponding to the three sums over index k, repeated at integer multiples of the revolution frequency. Figure 1 shows an idealized spectrum and the role played by the three frequency parameters $\square_r$, $\square_s$, and $\square_\square$. The group centered at $n\square_r \square \square_\square$ is referred to as the slow wave, and the group centered at $n\square_r + \square_\square$ is referred to as the fast wave. The asymmetry between the slow and fast waves is caused by nonzero $\square_\square$.

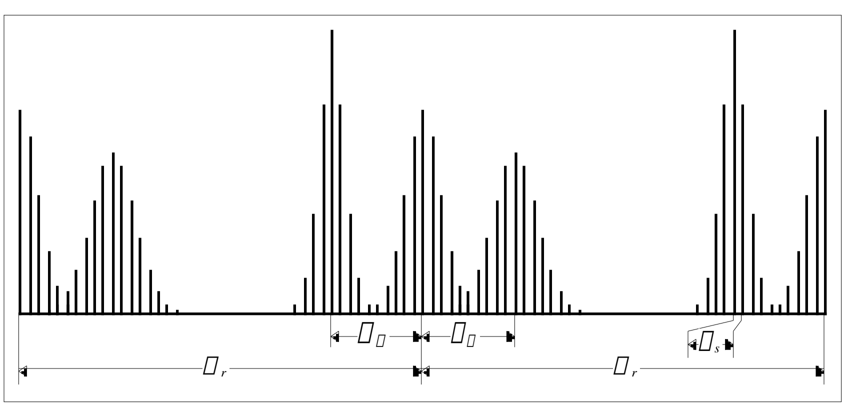

Figure 1

Estimation

The unknown quantities estimated from an observed Schottky spectrum are ν_β and ξ. The system used during the RHIC 2001 run used a peak-finding routine for tune estimation, and measurement of the FWHM of the slow and fast waves for chromaticity estimation. [Cameron, personal communication] Although this yielded accurate results, the Schottky spectra had to be time-averaged over periods on the order of tens of seconds to reduce estimate variance to acceptable levels. This averaging introduced an unacceptable delay in instrument response, since significant beam loss driven by bad values occurs on time scales on the order of seconds.

For effective operational control, it is important that estimates of ν_β and ξ are calculated quickly. Efficient estimation by numerically fitting a function of the form (1) to an observed spectrum is a difficult proposition, as application of this method to the RHIC Schottky system would require the evaluation of ~100 Bessel functions for each step of a χ^2 minimization. However, as long as the fast and slow waves are distinct from the central peak a computationally efficient method analogous to the statistical method of moments can be used.

Application of the method of moments hinges on approximating the modulation of the delta functions in both the slow and fast waves by functions of the form

(2) $$P(k) = I_k\left(\sigma^2\right) e^{-\sigma^2}.$$

This approximation should be good for sufficiently small σ_s, as the change in modulation with changing k will increasingly be dominated by change in the index of the Bessel function. $P(k)$ can be treated as a probability distribution function over discrete variable k, since $I_k\left(\sigma^2\right) e^{-\sigma^2} \geq 0$ for all k, and applying the Bessel function identity $\sum_{k=-\infty}^{\infty} I_k(z)\, t^k = e^{z(t+1/t)/2}$ with $t = 1$ gives $\sum_{k=-\infty}^{\infty} I_k\left(\sigma^2\right) e^{-\sigma^2} = 1$. The moment-generating function of $P(k)$ follows by application of the same identity:

(3) $$\begin{aligned} M(n) &= \sum_{k=-\infty}^{\infty} I_k\left(\sigma^2\right) e^{-\sigma^2} e^{nk} \\ &= e^{\sigma^2(\cosh(n)-1)} \end{aligned}$$

The first and second moments of $P(k)$ are zero and σ^2.

The RHIC Schottky system places the origin of the frequency axis at $\omega = n\omega_r$. The mth raw moment of a single observed slow or fast wave with this choice of origin is given by:

$$
(4)\qquad
\begin{aligned}
\mu_m &= \int d\omega \left(\omega - n\omega_r\right)^m I_k\left[\sigma_t^2\left(\omega \pm \left(\omega_\beta + \omega_\xi\right)\right)^2\right]\exp\left[-\sigma_t^2\left(\omega \pm \left(\omega_\beta + \omega_\xi\right)\right)^2\right]\delta\left(\omega - \left(\mp\omega_\beta + n\omega_r + k\omega_s\right)\right)\\
&= \sum_{k=-\infty}^{\infty} \left(\mp\omega_\beta + k\omega_s\right)^m I_k\left(\sigma_t^2\left(n\omega_r \pm \omega_\xi\right)^2\right)\exp\left(-\sigma_t^2\left(n\omega_r \pm \omega_\xi\right)^2\right)
\end{aligned}
$$

The normalized first and second raw moments of the slow and fast waves are therefore

$$
(5)\qquad
\begin{aligned}
\mu_1(fast) &= \omega_\beta \qquad \mu_2(fast) = \omega_\beta^2 + \omega_s^2\sigma_t^2\left(n\omega_r + \omega_\xi\right)^2\\
\mu_1(slow) &= -\omega_\beta \qquad \mu_2(slow) = \omega_\beta^2 + \omega_s^2\sigma_t^2\left(n\omega_r - \omega_\xi\right)^2,
\end{aligned}
$$

which lead to the estimators

$$
(6)\qquad \nu_\beta = \frac{\mu_1(fast) - \mu_1(slow)}{2\omega_r}, \quad \xi = \frac{\eta}{4n\nu_\beta\omega_r^2\omega_s^2\sigma_t^2}\left(\mu_2(fast) - \mu_2(slow)\right).
$$

Gaussian Approximation

As shown above, the first and second moments of a Schottky wave are zero and σ^2. These are identical to those of a Gaussian of standard deviation σ, suggesting the approximation

$$
(7)\qquad P(k) = I_k\left(\sigma^2\right)e^{-\sigma^2} \approx \frac{1}{\sqrt{2\pi}\sigma}e^{-k^2/2\sigma^2}.
$$

This can be derived explicitly from the integral definition of the modified Bessel function

$$
(8)\qquad I_k\left(\sigma^2\right) = \frac{1}{\pi}\int_0^{\pi}\cos(kt)\exp\left(\sigma^2\cos(t)\right)dt.
$$

For large σ^2, the $\cos(t)$ can be expanded to second order about $t = 0$, giving

$$
(9)\qquad I_k\left(\sigma^2\right) \approx \frac{1}{\sqrt{2\pi}\sigma}\exp\left(\sigma^2 - k^2/2\sigma^2\right)\left[\mathrm{Re}\left\{\mathrm{erf}\left(\frac{\pi\sigma^2 + ik}{\sqrt{2}\sigma}\right)\right\}\right].
$$

The value of the term in square brackets approaches one except for $|k| >> \sigma^2$, when its divergence is suppressed by the $-k^2/2\sigma^2$ term in the exponential. The continuous approximation and discrete $P(k)$ are plotted for several values of σ in Figure 2, showing excellent agreement.

Applying (7) to (1) gives an approximation of the Schottky spectrum composed of an infinite sum of Gaussian functions,

(10) $$F(\square)\ \square \underset{n=\square}{\square}\ A N\left(n\square_r + \square_\square, \square_t \square_s\left(n\square_r \square \square_\square\right)\right) + B N\left(n\square_r, \square_t \square_s n\square_r\right) + A N\left(n\square_r \square \square_\square, \square_t \square_s\left(n\square_r + \square_\square\right)\right)$$

where $N(\square,\square) = \exp\left(\square(\square \square\square)^2 \big/ 2\square^2\right) \big/ \sqrt{2\square}\,\square$.

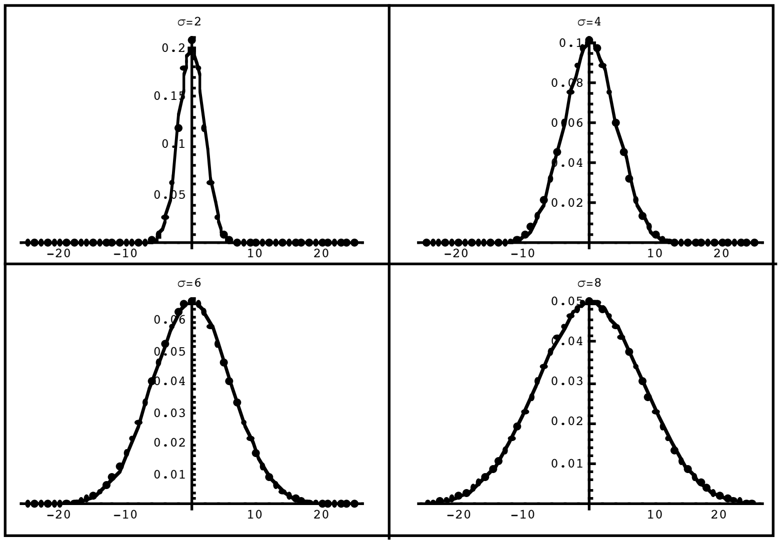

Figure 2 - P(k) and Gaussian approximation

Application

The quantity observed at RHIC is the log power spectrum, since the central peak is orders of magnitude stronger than the slow and fast waves. Equation 10 implies that all peaks in a log power spectrum should have quadratic form. Figure 3 shows an observed Schottky spectrum from the 2001 RHIC run, taken from the horizontal plane of the blue ring. Quadratic fits to the slow and fast waves clipped at -95 dBm are shown. The R^2 values for both fits are 0.98. The central peak was not fit because of coherent noise contamination at the revolution frequency.

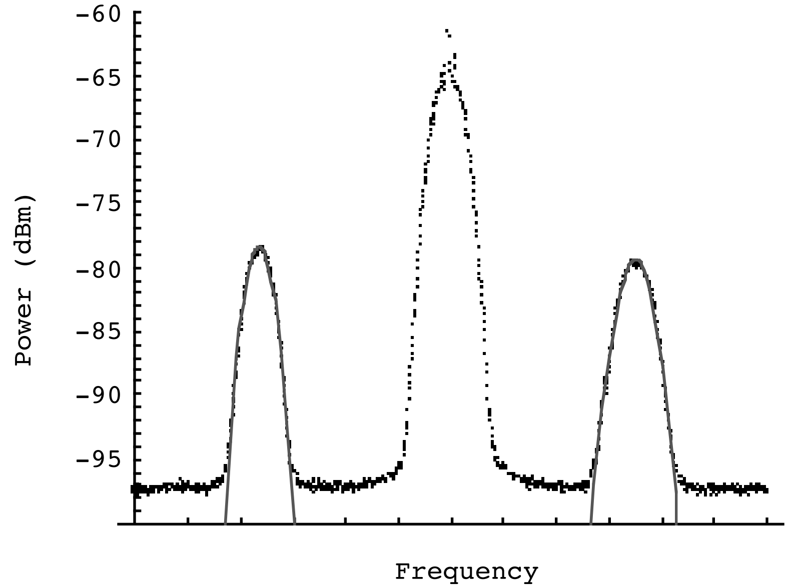

Figure 3 - Schottky spectrum (log scale) and quadratic fits

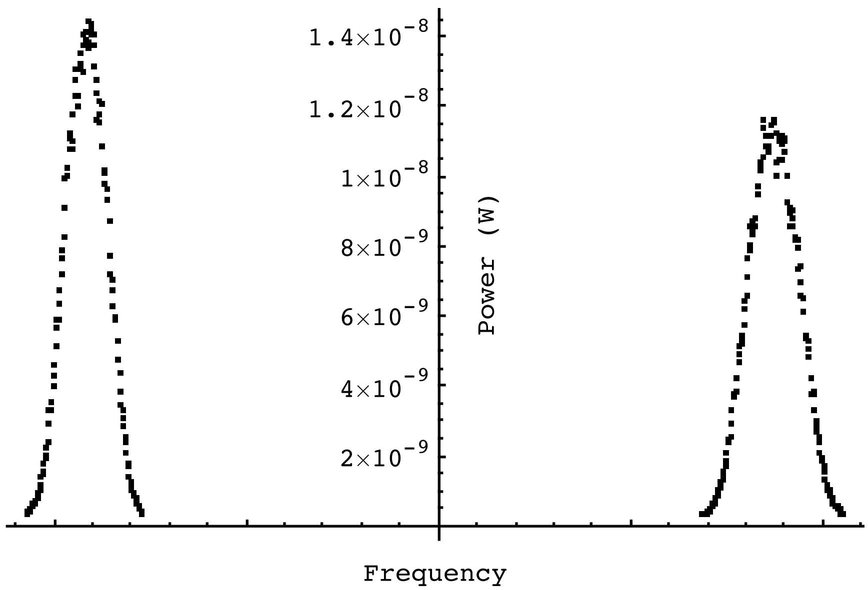

Figure 4 - Schottky spectrum (linear scale)

Figure 4 shows the fast and slow waves from the same Schottky spectrum, again clipped at -95 dBm, converted to a linear power scale. The table below gives the moments of each wave.

Moment	Slow Wave	Fast Wave
0	7.839 E-7	8.003 E-7
1	-0.115123 E6	0.109606 E6
2	1.26702 E10	1.26912 E10

The values of the zeroth moments agree to one part in fifty. The absolute values of the first moments are not precisely equal, indicating that the spectrum is slightly off-center. The value of $\square_{\square}$ derived from these moments is 0.228. Using values of 34 Hz for the RHIC synchrotron tune [Blaskiewicz], 3 nsec for bunch length, 78.125 kHz for revolution frequency [Gardner], and $2.069\ \mathrm{GHz}/78{,}125\ \mathrm{kHz} \,\square\, 25640$ [Cameron] for *n* gives -16.5 for $\square$, or -3.8 in conventional RHIC chromaticity units.

Works cited

Siemann, R. H. **Bunched Beam Diagnostics**, AIP Conf. Proc. 184(1) 430 (01 Apr 1989)
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