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An Emittance Exchanger

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1 Motivation

High field high energy proton colliders, such as a 50 TeV HECATEV with 12 Tesla dipoles, will experience radiation damping times of order one hour. Round beams will become flat. That is, while the emittances at the beginning of the store are approximately equal, $\epsilon_x \approx \epsilon_y$, the vertical emittance will shrink to an equilibrium value that is much smaller than the equilibrium horizontal emittance, $\epsilon_x >> \epsilon_y$. This is in general a "good thing" - doublet IR optics are simpler than triplet optics, for example - but it is inconvenient that the beam is not flat all the time.

Is it possible to include an "emittance exchanger" section in the transfer line from the injector to HECATEV, so that round beams going in become flat beams coming out? What is possible using skew quadrupoles in the transfer line, without violating Liouvilles theorem?

2 Normal modes - notation

Unfortunately, it is necessary to develop a somewhat bulky set of notation, in order to be explicit. A reader who is relatively disinterested in the details might wish to skip this section. A reader who is *really* disinterested might skip to the "Summary and Conclusions".

Here goes. The 4 physical coordinates at a reference point in a storage ring are written

$$Y = \begin{pmatrix} x_p \\ x'_p \\ y_p \\ y'_p \end{pmatrix} \tag{1}$$

where x_p and y_p are horizontal and vertical displacements, and x_p' and y_p' are horizontal and vertical angles. Normalized coordinates, written as

$$X = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} \tag{2}$$

are related to the physical coordinates through

$$X = \begin{pmatrix} G_x & 0 \\ 0 & G_y \end{pmatrix} Y \tag{3}$$

where, for example, the 2 by 2 matrix G_x is given by

$$G_x = \begin{pmatrix} 1/\sqrt{\beta_x} & 0 \\ \alpha_x/\sqrt{\beta_x} & \sqrt{\beta_x} \end{pmatrix} \tag{4}$$

The Twiss functions in this expression, β_x and α_x, are explicitly the *design* values at the reference point.

Linearized motion is represented by the one turn matrix T, so that in going from turn n to $n+1$

$$X_{n+1} = T X_n \tag{5}$$

In general, the 4 by 4 matrix T is modified from its design value by coupling (and other) errors. Edwards and Teng [1] have shown that matrices U and V always exist such that

$$T = VUV^{-1} \tag{6}$$

where U is block diagonal in unimodular 2 by 2 matrices A and B

$$U = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \tag{7}$$

The matrix V is conveniently written in component form as [2]

$$V = \begin{pmatrix} c & 0 & sd & se \\ 0 & c & sf & sg \\ -sg & se & c & 0 \\ sf & -sd & 0 & c \end{pmatrix} \tag{8}$$

where the components are related through

$$\begin{array}{rcl} c & = & \cos(\psi) \\ s & = & \sin(\psi) \\ dg - fe & = & 1 \end{array} \tag{9}$$

The angle ψ lies in the range

$$-\pi/4 \;\;\leq\;\; \psi \;\;\leq\;\; \pi/4 \tag{10}$$

It only deviates significantly from zero when the difference in the eigentunes, $Q_1 - Q_2$, approaches its minimum value - that is, when the tune diagonal is approached and the beams become strongly coupled [2].

The linear motion of a single test particle has now been solved. If the normalized displacement on turn 0 is X_0, then on turn n it is

$$X_n \;\;=\;\; VU^n \, (V^{-1} X_0) \tag{11}$$

This motion is easily visualized when this motion is projected onto the (normalized) real space plane (x, y), for the special cases shown in Figure 1.

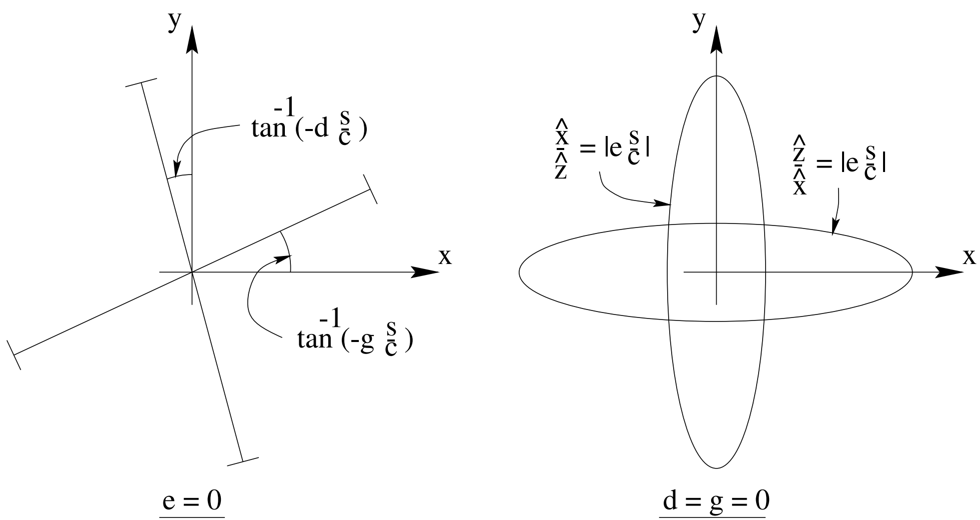

Figure 1: Normal modes, projected onto the (normalized) real space plane, (x, y), for two special cases.

3 Eigenaction and eigenemittance

The 4-D motion is conveniently described by defining eigenactions J_1 and J_2, and eigenphases ϕ_1 and ϕ_2, such that

$$\begin{array}{lcll} x & = & \sqrt{2J_1}c\, c_1 & + \; \sqrt{2J_2}s\,(dc_2 + es_2) \\ x' & = & \sqrt{2J_1}c\, s_1 & + \; \sqrt{2J_2}s\,(fc_2 + gs_2) \\ y & = & \sqrt{2J_1}s\,(-gc_1 + es_1) & + \; \sqrt{2J_2}c\, c_1 \\ y' & = & \sqrt{2J_1}s\,(\;\;fc_1 - ds_1) & + \; \sqrt{2J_2}c\, s_1 \end{array} \tag{12}$$

where terms like c_1 and s_1 are shorthand for trigonometric terms like

$$c_1 = \cos(2\pi Q_1 n + \phi_1) \tag{13}$$

et cetera. The eigenactions and eigenphases are found for a test particle with initial conditions X_0 by comparing Equation 12 with Equations 8 and 11. The eigenaction assumes its familiar role in the absence of coupling, when $s = 0$ and $c = 1$. For example, the decoupled eigenaction on any turn is then simply calculated as

$$J_1 = \frac{1}{2}(x^2 + x'^2) \tag{14}$$

In general, the eigenemittance of an ensemble of test particles is given by

$$\epsilon_1 = \int_0^\infty J_1 \rho(J_1)\, dJ_1 \tag{15}$$

where $\rho(J_1)$ is the probability distribution of the eigenaction. This, too, becomes familiar in the absence of coupling, when the root mean square horizontal beam size is given by

$$< x_p^2 >^{1/2} = \sqrt{\epsilon_1 \beta_x} \tag{16}$$

where angle brackets imply an average over both action and phase, J_1 and ϕ_1.

4 A transfer line with skew quads

The question now becomes "How do eigenemittances ϵ_1 and ϵ_2, coming out of a general transfer line with adjustable design parameters, transform into ϵ_x and ϵ_y in the following ring?" This is answered by first considering how the eigenactions are transformed.

The normal modes in an injector may be projected through a transfer line into a second ring, using the same V matrix notation as above. Skew quadrupoles deliberately inserted in the transfer line may be used to manipulate the matrix components, within the constraints set by Equations 9 and 10. Without loss of generality with respect to the predictions of ultimate emittance exchanger performance, derived below, it may be assumed that the second ring is perfect and normal, so that *its* normal modes are regular and erect, with eigenemittances ϵ_x and ϵ_y.

In that case, the horizontal and vertical actions for a test particle parameterized by $(J_1, \phi_1, J_2, \phi_2)$ are simply given by

$$J_x = \frac{1}{2}(x^2 + x'^2) \tag{17}$$

$$J_y = \frac{1}{2}(y^2 + y'^2) \tag{18}$$

After contemplating substituting Equation 12 into the right hand sides of these two equations, it becomes clear that the new actions J_x and J_y depend on the old eigenphases ϕ_1 and ϕ_2. After averaging over the eigenphases for all test particles with fixed (J_1, J_2), and assuming that (ϕ_1, ϕ_2) are smoothly and independently distributed, then

$$\begin{aligned} < J_x > &= c^2 J_1 \ + \ s^2 J_2 \frac{1}{2}(d^2 + e^2 + f^2 + g^2) \\ < J_y > &= s^2 J_1 \frac{1}{2}(d^2 + e^2 + f^2 + g^2) \ + \ c^2 J_2 \end{aligned} \tag{19}$$

Clearly, if the values (d, e, f, g) are large, then the actions (and emittances) will be significantly blown up.

Referring to Equation 9, it is easy to show that

$$\frac{1}{2}(d^2 + e^2 + f^2 + g^2) \ \geq \ 1 \tag{20}$$

where the minimum value of 1 occurs when

$$\begin{aligned} d &= g = 1 \\ e &= f = 0 \end{aligned} \tag{21}$$

This corresponds to one of the special cases in Figure 1, where both eigenmodes are tilted by the same angle ψ. Evidently, Equation 21 defines what is meant by a "well matched" transfer line, so far as skew quadrupoles are concerned.

Assuming that the transfer line is well matched (Equation 21 is true), then Equation 19 becomes

$$< J_x > \ = \ c^2 J_1 \ + \ s^2 J_2 \tag{22}$$

$$< J_y > \ = \ s^2 J_1 \ + \ c^2 J_2 \tag{23}$$

Integrating over J_1 and J_2, this simply becomes

$$\begin{aligned} \epsilon_x &= \cos^2(\psi)\, \epsilon_1 \ + \ \sin^2(\psi)\, \epsilon_2 \\ \epsilon_y &= \sin^2(\psi)\, \epsilon_1 \ + \ \cos^2(\psi)\, \epsilon_2 \end{aligned} \tag{24}$$

This describes the optimal design performance of a matched transfer line, acting as emittance exchanger.

5 Summary and Conclusions

A transfer line that incorporates skew quadrupoles can be used as an "emittance exchanger", in order to modify incoming emittances ϵ_1 and ϵ_2 to outgoing emittances ϵ_x and ϵ_y. Inspection of Equation 24 reveals that, under well tuned conditions, the emittance sum remains unchanged:

$$\epsilon_x + \epsilon_y \quad = \quad \epsilon_1 + \epsilon_2 \tag{25}$$

Unfortunately, Equation 24 also reveals that the beams can be made rounder, but not flatter:

$$1 \quad \leq \quad \frac{\epsilon_x}{\epsilon_y} \quad \leq \quad \frac{\epsilon_1}{\epsilon_2} \tag{26}$$

where it is assumed that $\epsilon_1 \geq \epsilon_2$. Such an emittance exchanger is no help in making HECATEV beam flatter at injection - without blowing up the total emittance. An emittance exchanger may have other potential uses, perhaps in electron or muon transfer lines.

References

[1] D. Edwards and L. Teng, IEEE Trans. Nucl. Sci., NS-20, No. 3, 1973.

[2] S. Peggs, "Coupling and Decoupling in Storage Rings", IEEE Trans. Nucl. Sci., NS-30, No. 4, p.2460, Santa Fe PAC, August 1983.