KINETIC DESCRIPTION OF ELECTRON-PROTON (e-p) INSTABILITY IN HIGH-INTENSITY LINACS AND STORAGE RINGS*

by

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Recent Publications on the e-p Instability

- "Kinetic Description of Electron-Proton Instability in High-Intensity Proton Linacs and Storage Rings Based on the Vlasov-Maxwell Equations," R. C. Davidson, H. Qin, P. H. Stoltz, and T. -S. Wang, *Physical Review Special Topics on Accelerators and Beams* **2**, 054401 (1999).

- "Vlasov-Maxwell Description of Electron-Ion Two-Stream Instability in High-Intensity Linacs and Storage Rings," R. C. Davidson, H. Qin, and T. -S. Wang, *Physics Letters A* **252**, 213 (1999).

- "Kinetic Description of the Electron-Proton Instability in High-Intensity Linacs and Storage Rings," R. C. Davidson, H. Qin, W. W. Lee, and T. -S. Wang, *Proceedings of the 1999 Particle Accelerator Conference,* in press (1999).

- "Multispecies Nonlinear Perturbative Particle Simulation of Intense Charged Particle Beam," H. Qin, R. C. Davidson, and W. W. Lee, *Proceedings of the 1999 Particle Accelerator Conference,* in press (1999).

Theoretical Model and Assumptions

- Consider high-intensity ion beam with distribution function $f_b(\mathbf{x}, \mathbf{p}, t)$, characteristic radius r_b, and axial momentum $\gamma_b m_b \beta_b c$, propagating in z-direction through background population of electrons with distribution function $f_e(\mathbf{x}, \mathbf{p}, t)$.

- Ions have high directed axial velocity $V_b = \beta_b c$, whereas electrons are nonrelativistic and stationary in the laboratory frame with $\int d^3p p_z f_e(\mathbf{x}, \mathbf{p}, t) \simeq 0$.

- Ion beam is treated as continuous in the z-direction, and applied transverse focusing force if modeled by

$$\mathbf{F}^b_{foc} = -\gamma_b m_b \omega^2_{\beta b} \mathbf{x}_\perp$$

in the smooth-beam approximation, where $\mathbf{x}_\perp = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y$ is transverse displacement from beam axis.

Theoretical Model and Assumptions

- For ion-rich beam, the space-charge force on an electron, $\mathbf{F}_e^s = e\nabla\phi$, provides transverse confinement of the electrons by the electrostatic potential $\phi(\mathbf{x}, t)$.

- Ion motion in the beam frame is assumed to be nonrelativistic, with

$$|p_x| \ , |p_y| \ |\delta p_z| \ll \gamma_b m_b \beta_b c$$

  where $\delta p_z = p_z - \gamma_b m_b \beta_b c$, and $\gamma_b m_b \beta_b c$ is the directed axial momentum.

- Allow arbitrary space-charge intensity consistent with radial confinement of the ions and

$$\nu_B \equiv \frac{Z_b^2 e^2 N_b}{m_b c^2} \ll \gamma_b$$

  where $N_b = \int dxdyn_b$ is the number of ions per unit axial length.

Theoretical Model and Assumptions

- Analysis is carried out in the electrostatic approximation where the self-generated electric field is

$$\mathbf{E}^s(\mathbf{x},t) = -\nabla\phi(\mathbf{x},t)$$

- The electrostatic potential $\phi(x,y,z,t)$ is determined self-consistently from Poisson's equation

$$\nabla^2\phi = -4\pi e(Z_b n_b - n_e)$$

where $n_b(\mathbf{x},t) = \int d^3p f_b(\mathbf{x},\mathbf{p},t)$ and $n_e = \int d^3p f_e(\mathbf{x},\mathbf{p},t)$ are the ion and electron number densities.

- Assume that the ion axial velocity profile $V_{zb}(\mathbf{x},t) \simeq \beta_b c$ is approximately uniform over the beam cross section. The self-generated magnetic field

$$\mathbf{B}^s(\mathbf{x},t) = \nabla A_z(\mathbf{x},t) \times \hat{\mathbf{e}}_z$$

is determined from

$$\nabla^2 A_z = -4\pi Z_b e \beta_b n_b$$

where the electrons are assumed to carry zero axial current in the laboratory frame.

Theoretical Model and Assumptions

- Under equilibrium conditions ($\partial/\partial t = 0$), treat the ion and electron properties as spatially uniform in the z-direction ($\partial/\partial z = 0$).

- In the stability analysis, assume small-amplitude perturbations with $z-$ and $t-$variations of the form
$$\exp(ik_z z - i\omega t)$$
where $Im\omega > 0$ corresponds to instability (temporal growth), and $k_z = 2\pi n/L$ is the axial wavenumber, where n is an integer, and L is the axial periodicity length of the perturbation.

- Stability analysis assumes perturbations with sufficiently high frequency ω and long axial wavelength $2\pi/k_z$ that
$$k_z^2 r_b^2 \ll 1$$
$$\left|\frac{\omega}{k_z} - \beta_b c\right| \gg v_{Tbz}$$
$$\left|\frac{\omega}{k_z}\right| \gg v_{Tez}$$
where $v_{Tbz} = (2T_{bz}/\gamma_b m_b)^{1/2}$ and $v_{Tez} = (2T_{ez}/m_e)^{1/2}$ are the characteristic axial thermal speeds.

Theoretical Model and Assumptions

The assumption of high-frequency perturbations with long axial wavelength leads to several simplifications in the analysis of the Vlasov-Maxwell equations.

- The three-dimensional Laplacian ∇^2 is approximated by

$$\nabla^2 \simeq \nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

- The perturbed axial forces on the electrons and ions, e.g.,

$$\delta \mathbf{F}_e = e\frac{\partial}{\partial z}\delta\phi\hat{\mathbf{e}}_z \quad \text{and} \quad \delta \mathbf{F}_b = -Z_b e\frac{\partial}{\partial z}\delta\phi\hat{\mathbf{e}}_z$$

  are treated as small in comparison with the transverse forces (i.e., neglect the effects of resonant Landau damping in axial velocity space v_z).

- Therefore, we describe the evolution of system in terms of the reduced distribution functions

$$F_b(\mathbf{x}, \mathbf{p}_\perp, t) = \int dp_z f_b(\mathbf{x}, \mathbf{p}, t)$$

$$F_e(\mathbf{x}, \mathbf{p}_\perp, t) = \int dp_z f_e(\mathbf{x}, \mathbf{p}, t)$$

Nonlinear Vlasov-Maxwell Equations

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- In the context of these assumptions, the electron distribution $F_e(\mathbf{x}, \mathbf{p}_\perp, t)$ evolves nonlinearly according to

$$\left\{ \frac{\partial}{\partial t} + \frac{\mathbf{p}_\perp}{m_e} \cdot \frac{\partial}{\partial \mathbf{x}_\perp} + e\nabla_\perp \phi \cdot \frac{\partial}{\partial \mathbf{p}_\perp} \right\} \times F_e(\mathbf{x}, \mathbf{p}_\perp, t) = 0$$

where $-e$ is the electron charge, and $\nabla_\perp \equiv \hat{\mathbf{e}}_x \partial/\partial x + \hat{\mathbf{e}}_y \partial/\partial y$ is the perpendicular gradient.

- For the ions, $\mathbf{v} \cdot \partial/\partial \mathbf{x} \simeq (\mathbf{p}_\perp/\gamma_b m_b) \cdot \partial/\partial \mathbf{x}_\perp + V_b \partial/\partial z$, and the nonlinear Vlasov equation for $F_b(\mathbf{x}, \mathbf{p}_\perp, t)$ becomes

$$\left\{ \frac{\partial}{\partial t} + V_b \frac{\partial}{\partial z} + \frac{\mathbf{p}_\perp}{\gamma_b m_b} \cdot \frac{\partial}{\partial \mathbf{x}_\perp} \right.$$

$$\left. - \left(\gamma_b m_b \omega_{\beta b}^2 \mathbf{x}_\perp + Z_b e \nabla_\perp \psi \right) \cdot \frac{\partial}{\partial \mathbf{p}_\perp} \right\} F_b(\mathbf{x}, \mathbf{p}_\perp, t) = 0$$

- Here, $+Z_b e$ is the ion charge, and $\psi(\mathbf{x}, t)$ is the combined potential defined by

$$\psi(\mathbf{x}, t) \equiv \phi(\mathbf{x}, t) - \beta_b A_z(\mathbf{x}, t)$$

Nonlinear Vlasov-Maxwell Equations

- The electrostatic potential $\phi(\mathbf{x},t)$ and combined potential $\psi(\mathbf{x},t) = \phi(\mathbf{x},t) - \beta_b A_z(\mathbf{x},t)$ are determined self-consistently from

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -4\pi e\left(Z_b \int d^2p F_b - \int d^2p F_e\right)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = -4\pi e\left(\frac{Z_b}{\gamma_b^2} \int d^2p F_b - \int d^2p F_e\right)$$

- In Maxwell's equations for $\phi(\mathbf{x},t)$ and $\psi(\mathbf{x},t)$

$$n_b(\mathbf{x},t) = \int d^2p F_b(\mathbf{x},\mathbf{p}_\perp,t)$$

$$n_e(\mathbf{x},t) = \int d^2p F_e(\mathbf{x},\mathbf{p}_\perp,t)$$

  are the ion and electron particle densities, respectively.

- The thin-beam approximation and $k_z^2 r_b^2 \ll 1$ have been used to approximate $\nabla^2 \simeq \nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

Nonlinear Vlasov-Maxwell Equations

- Assume that a perfectly conducting cylindrical wall is located at radius $r = r_w$, where $r = (x^2 + y^2)^{1/2}$. Impose the requirement that

$$[E^s_\theta]_{r=r_w} = [E^s_z]_{r=r_w} = [B^s_r]_{r=r_w} = 0$$

- In terms of the potentials $\phi(\mathrm{x}, t)$ and $\psi(\mathrm{x}, t)$, this gives

$$\phi(r = r_w, \theta, z, t) = 0$$

$$\psi(r = r_w, \theta, z, t) = 0$$

where the constant values of the potentials at $r = r_w$ have been taken equal to zero without loss of generality.

Equilibrium Vlasov-Maxwell Equations

- Under quasisteady conditions, examine solutions to nonlinear Vlasov-Maxwell equations with

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \theta} = \frac{\partial}{\partial z} = 0$$

- Vlasov-Maxwell equations support broad range of equilibrium solutions for the beam ions and background electrons of the general form

$$\begin{aligned} F_b^0 &= F_b^0(H_{\perp b}) \\ F_e^0 &= F_e^0(H_{\perp e}) \end{aligned}$$

- Here, $H_{\perp b}$ and $H_{\perp e}$ are the single-particle Hamiltonians defined by

$$\begin{aligned} H_{\perp b} &= \frac{1}{2\gamma_b m_b}\mathbf{p}_\perp^2 + \frac{1}{2}\gamma_b m_b \omega_{\beta b}^2 r^2 + Z_b e[\psi^0(r) - \hat{\psi}^0] \\ H_{\perp e} &= \frac{1}{2m_e}\mathbf{p}_\perp^2 - e[\phi^0(r) - \hat{\phi}^0] \end{aligned}$$

where $r = (x^2 + y^2)^{1/2}$, and the constants $\hat{\phi}^0 \equiv \phi^0(r = 0)$ and $\hat{\psi}^0 \equiv \psi^0(r = 0)$ are the on-axis values of the potentials.

Equilibrium Vlasov-Maxwell Equations

- For specified distribution functions $F_b^0(H_{\perp b})$ and $F_e^0(H_{\perp e})$, the equilibrium potentials $\phi^0(r)$ and $\psi^0(r)$ are determined self-consistently from

$$
\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\phi^0(r) &= -4\pi e[Z_b n_b^0(r) - n_e^0(r)] \\
\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\psi^0(r) &= -4\pi e\left[\frac{Z_b}{\gamma_b^2}n_b^0(r) - n_e^0(r)\right]
\end{aligned}
$$

where $n_b^0(r)$ and $n_e^0(r)$ are the ion and electron density profiles

$$
\begin{aligned}
n_b^0(r) &= \int d^2p F_b^0(H_{\perp b}) \\
n_e^0(r) &= \int d^2p F_e^0(H_{\perp e})
\end{aligned}
$$

- Maxwell's equations for $\phi^0(r)$ and $\psi^0(r)$ are generally *nonlinear*.

Equilibrium with Step-Function Density Profiles

- A simple class of equilibrium distribution functions $F_b^0(H_{\perp b})$ and $F_e^0(H_{\perp e})$, which correspond to overlapping *step-function* density profiles for the beam ions and background electrons, is given by

$$F_b^0(H_{\perp b}) = \frac{\widehat{n}_b}{2\pi\gamma_b m_b}\delta(H_{\perp b} - \widehat{T}_{\perp b})$$

$$F_e^0(H_{\perp e}) = \frac{\widehat{n}_e}{2\pi m_e}\delta(H_{\perp e} - \widehat{T}_{\perp e})$$

where $\widehat{n}_b$, $\widehat{n}_e$, $\widehat{T}_{\perp b}$, and $\widehat{T}_{\perp e}$ are positive constants.

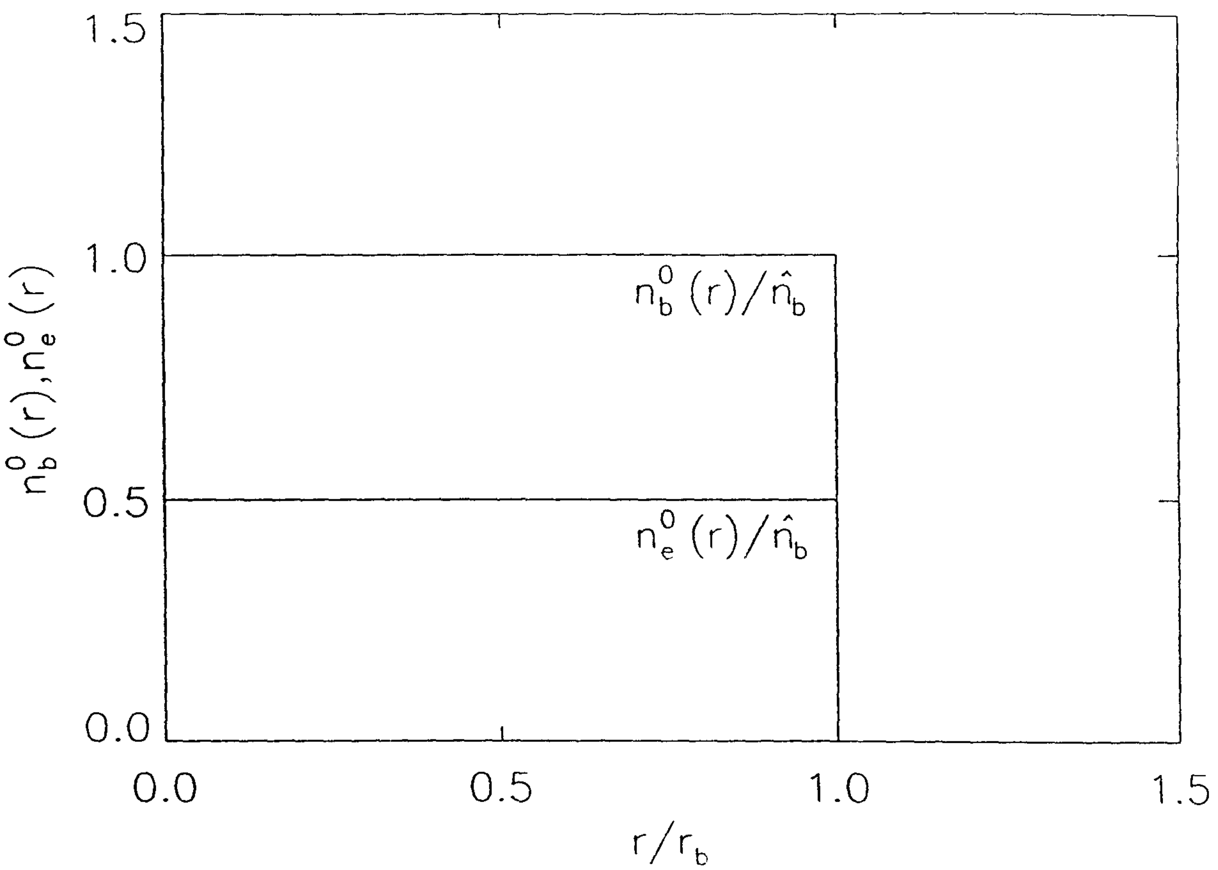
- Some straightforward algebraic manipulation shows that the corresponding density profiles are

$$n_b^0(r) = \begin{cases} \widehat{n}_b = const., & 0 \leq r < r_b \\ 0\ , & r_b < r \leq r_w \end{cases}$$

and

$$n_e^0(r) = \begin{cases} \widehat{n}_e \equiv f Z_b \widehat{n}_b = const., & 0 \leq r < r_b \\ 0\ , & r_b < r \leq r_w \end{cases}$$

where $f \equiv \widehat{n}_e / Z_b \widehat{n}_b$ is the fractional charge neutralization.


1.5
1.0
0.5
0.0
$n_b^0(r), n_e^0(r)$
$n_b^0(r)/\hat{n}_b$
$n_e^0(r)/\hat{n}_b$
0.0
0.5
1.0
1.5
r/r_b

Equilibrium with Step-Function Density Profiles

- Introduce the ion plasma frequency-squared defined by

$$\widehat{\omega}_{pb}^2 \equiv \frac{4\pi \widehat{n}_b Z_b^2 e^2}{\gamma_b m_b} = \frac{4 N_b Z_b^2 e^2}{\gamma_b m_b r_b^2}$$

where $N_b = \pi \widehat{n}_b r_b^2$ is the number of beam ions per unit axial length.

- Equilibrium analysis shows that the beam radius r_b is related to $\widehat{T}_{\perp b}$, $\widehat{T}_{\perp e}$, $\widehat{\omega}_{pb}^2$, etc., by the equilibrium constraint conditions

$$\left[\omega_{\beta b}^2 - \frac{1}{2}\left(\frac{1}{\gamma_b^2} - f\right)\widehat{\omega}_{pb}^2\right] r_b^2 = \frac{2\widehat{T}_{\perp b}}{\gamma_b m_b}$$

$$\frac{1}{2}\frac{\gamma_b m_b}{Z_b m_e}\Big(1 - f\Big)\widehat{\omega}_{pb}^2\, r_b^2 = \frac{2\widehat{T}_{\perp e}}{m_e}$$

- The coefficients of r_b^2 in the above constraint conditions will be recognized as the *depressed* betatron frequencies

$$\widehat{\nu}_b^2 \text{ and } \widehat{\nu}_e^2$$

for transverse particle motions, including self-field effects.

Equilibrium with Step-Function Density Profiles

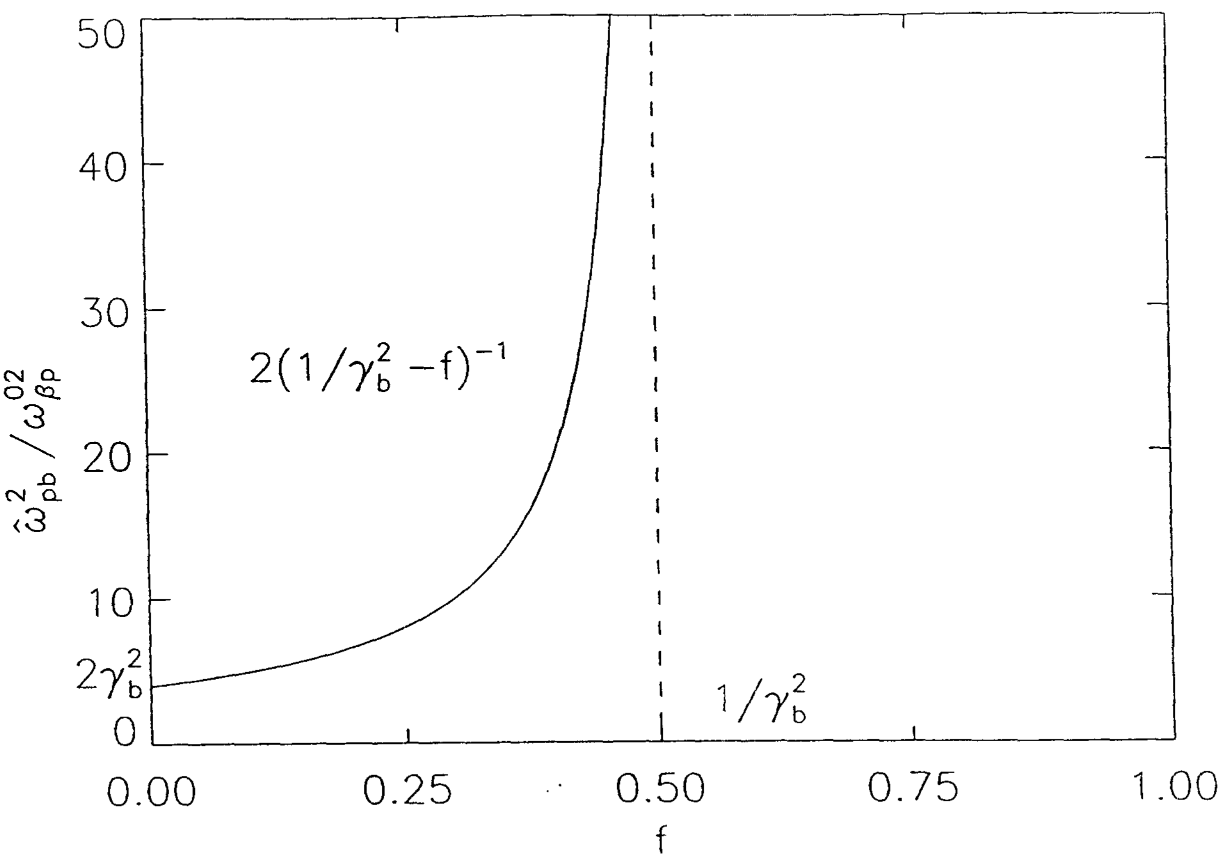
- Examine equilibrium constraint conditions for $\hat{T}_{\perp b} \geq 0$ and $\hat{T}_{\perp e} \geq 0$.

- Can show that both the ions and electrons are radially confined provided

$$f < 1$$

$$\frac{1}{2}\frac{\hat{\omega}_{pb}^2}{\omega_{\beta b}^2}\left(\frac{1}{\gamma_b^2} - f\right) < 1$$

which place restrictions on the allowed values of fractional charge neutralization f, and normalized beam intensity $\hat{\omega}_{pb}^2/\omega_{\beta b}^2$.



50
40
30
20
10
0
$2\gamma_b^2$
$\hat{\omega}_{pb}^2 / \omega_{\beta p}^{02}$
$2(1/\gamma_b^2 - f)^{-1}$
$1/\gamma_b^2$
0.00
0.25
0.50
0.75
1.00
f

Thermal Equilibrium with Diffuse Density Profiles

- Many choices of equilibrium distributions $F_b^0(H_{\perp b})$ and $F_e^0(H_{\perp e})$ are possible. As another example, consider

$$F_b^0(H_{\perp b}) = \frac{\widehat{n}_b}{(2\pi\gamma_b m_b T_{\perp b})} \exp\left(-\frac{H_{\perp b}}{T_{\perp b}}\right)$$

$$F_b^0(H_{\perp e}) = \frac{\widehat{n}_e}{(2\pi m_e T_{\perp e})} \exp\left(-\frac{H_{\perp e}}{T_{\perp e}}\right)$$

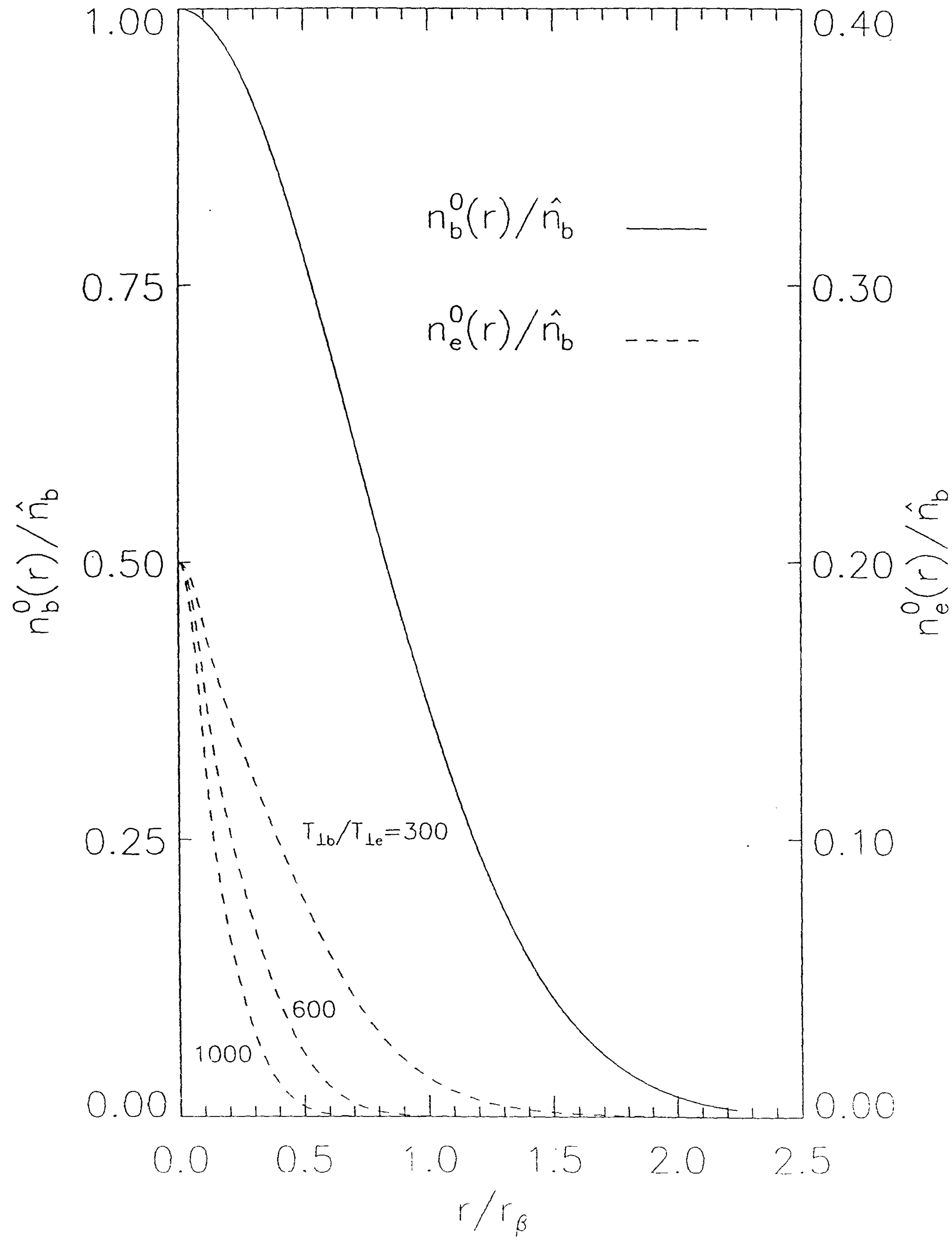
where $\widehat{n}_b$, $\widehat{n}_e$, $T_{\perp b}$, and $T_{\perp e}$ are positive constants.

- The corresponding equilibrium density profiles are

$$n_b^0(r) = \widehat{n}_b \exp\left\{-\frac{1}{T_{\perp b}}\left(\frac{1}{2}\gamma_b m_b \omega_{\beta b}^2 r^2 + Z_b e[\psi^0(r) - \widehat{\psi}^0]\right)\right\}$$

$$n_e^0(r) = \widehat{n}_e \exp\left\{\frac{e}{T_{\perp e}}[\phi^0(r) - \widehat{\phi}^0]\right\}$$

- The potentials $\psi^0(r)$ and $\phi^0(r)$ must be determined numerically from the corresponding Maxwell equations, which are highly nonlinear.



$n_b^0(r)/\hat{n}_b$
$n_e^0(r)/\hat{n}_b$
$T_{\perp b}/T_{\perp e}=300$
600
1000
1.00
0.75
0.50
0.25
0.00
0.40
0.30
0.20
0.10
0.00
0.0
0.5
1.0
1.5
2.0
2.5
r/r_β

Linearized Vlasov-Maxwell Equations

- Express all quantities in the nonlinear Vlasov-Maxwell equations as an equilibrium value plus a perturbation, e.g., $F_b(\mathbf{x},\mathbf{p}_\perp,t) = F_b^0(H_{\perp b}) + \delta F_b(\mathbf{x},\mathbf{p}_\perp,t)$, $\psi(\mathbf{x},t) = \psi^0(r) + \delta\psi(\mathbf{x},t)$, etc.

- For small-amplitude perturbations, the linearized Vlasov equation for the ions becomes

$$\left\{\frac{\partial}{\partial t} + V_b\frac{\partial}{\partial z} + \frac{\mathbf{p}_\perp}{\gamma_b m_b}\cdot\frac{\partial}{\partial \mathbf{x}_\perp}\right.$$

$$\left. - \left[\gamma_b m_b \omega_{\beta b}^2 + \frac{Z_b e}{r}\frac{\partial}{\partial r}\psi^0(r)\right]\mathbf{x}_\perp\cdot\frac{\partial}{\partial \mathbf{p}_\perp}\right\}\delta F_b(\mathbf{x},\mathbf{p}_\perp,t)$$

$$= \frac{Z_b e}{\gamma_b m_b}\mathbf{p}_\perp\cdot\nabla_\perp\delta\psi(\mathbf{x},t)\frac{\partial}{\partial H_{\perp b}}F_b^0(H_{\perp b})$$

Linearized Vlasov-Maxwell Equations

- Similarly, the linearized Vlasov equation for the electrons is given by

$$\left\{\frac{\partial}{\partial t}+\frac{\mathbf{p}_\perp}{m_e}\cdot\frac{\partial}{\partial \mathbf{x}_\perp}+\frac{e}{r}\frac{\partial}{\partial r}\phi^0(r)\mathbf{x}_\perp\cdot\frac{\partial}{\partial \mathbf{p}_\perp}\right\}$$

$$\times \quad \delta F_e(\mathbf{x},\mathbf{p}_\perp,t)$$

$$= \quad -\frac{e}{m_e}\mathbf{p}_\perp\cdot\nabla_\perp\delta\phi(\mathbf{x},t)\frac{\partial}{\partial H_{\perp e}}F_e^0(H_{\perp e})$$

- Linearized Vlasov-Maxwell equations are valid for small-amplitude perturbations about general choice of equilibrium distribution functions $F_b^0(H_{\perp b})$ and $F_e^0(H_{\perp e})$.

Linearized Vlasov-Maxwell Equations

- The perturbed potentials $\delta\psi(\mathbf{x},t)$ and $\delta\phi(\mathbf{x},t)$ are determined self-consistently in terms of the perturbed distribution functions from the Maxwell equations

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\delta\psi = -4\pi e\left(\frac{Z_b}{\gamma_b^2}\int d^2p\delta F_b - \int d^2p\delta F_e\right)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\delta\phi = -4\pi e\left(Z_b\int d^2p\delta F_b - \int d^2p\delta F_e\right)$$

- In the linearized Vlasov equations for $\delta F_b(\mathbf{x},\mathbf{p}_\perp,t)$ and $\delta F_e(\mathbf{x},\mathbf{p}_\perp,t)$, it is important to recognize that the differential operator

$$\{\cdots\} = \frac{d}{dt'}$$

corresponds to the *total time derivative* following the particle motion in the total equilibrium (applied plus self-generated) field configuration.

Linearized Vlasov-Maxwell Equations

- For amplifying perturbations, we integrate the linearized Vlasov equations from $t' = -\infty$, where the perturbations are negligibly small, up to the present time $t' = t$, when the particle orbits $\mathbf{x}'(t')$ and $\mathbf{p}'_\perp(t')$ pass through the phase-space point $(\mathbf{x}, \mathbf{p}_\perp)$, i.e.,

$$\begin{aligned} \mathbf{x}'(t' = t) &= \mathbf{x} \\ \mathbf{p}'_\perp(t' = t) &= \mathbf{p}_\perp \end{aligned}$$

- This gives for the perturbed distribution functions

$$\delta F_b(\mathbf{x}, \mathbf{p}_\perp, t) = Z_b e \frac{\partial}{\partial H_{\perp b}} F_p^0(H_{\perp b}) \int_{-\infty}^{t} dt' \frac{\mathbf{p}'_\perp}{\gamma_b m_b} \cdot \nabla'_\perp \delta\psi(\mathbf{x}', t')$$

$$\delta F_e(\mathbf{x}, \mathbf{p}_\perp, t) = -e \frac{\partial}{\partial H_{\perp e}} F_e^0(H_{\perp e}) \int_{-\infty}^{t} dt' \frac{\mathbf{p}'_\perp}{m_e} \cdot \nabla'_\perp \delta\phi(\mathbf{x}', t')$$

where use has been made of $dH'_{\perp b}/dt' = 0 = dH'_{\perp e}/dt'$.

Linearized Vlasov-Maxwell Equations

- The 'primed' orbits for the beam ions solve $z'(t') = z + V_b(t' - t)$ and

$$\frac{d}{dt'}\mathbf{x}'_\perp(t') = \frac{1}{\gamma_b m_b}\mathbf{p}'_\perp(t')$$

$$\frac{d}{dt'}\mathbf{p}'_\perp(t') = -\gamma_b m_b \omega_{\beta b}^2 \mathbf{x}'_\perp(t') - \frac{Z_b e}{r'}\frac{\partial \psi^0(r')}{\partial r'}\mathbf{x}'_\perp(t')$$

where $r'^2(t') = x'^2(t') + y'^2(t')$. Similarly, the 'primed' orbits for the background electrons solve $z'(t') = z$, and

$$\frac{d}{dt'}\mathbf{x}'_\perp(t') = \frac{1}{m_e}\mathbf{p}'_\perp(t')$$

$$\frac{d}{dt'}\mathbf{p}'_\perp(t') = \frac{e}{r'}\frac{\partial \phi^0(r')}{\partial r'}\mathbf{x}'_\perp(t')$$

where $\mathbf{x}'_\perp(t' = t) = \mathbf{x}_\perp$ and $\mathbf{p}'_\perp(t' = t) = \mathbf{p}_\perp$.