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Calculation of Longitudinal Space Charge Impedance

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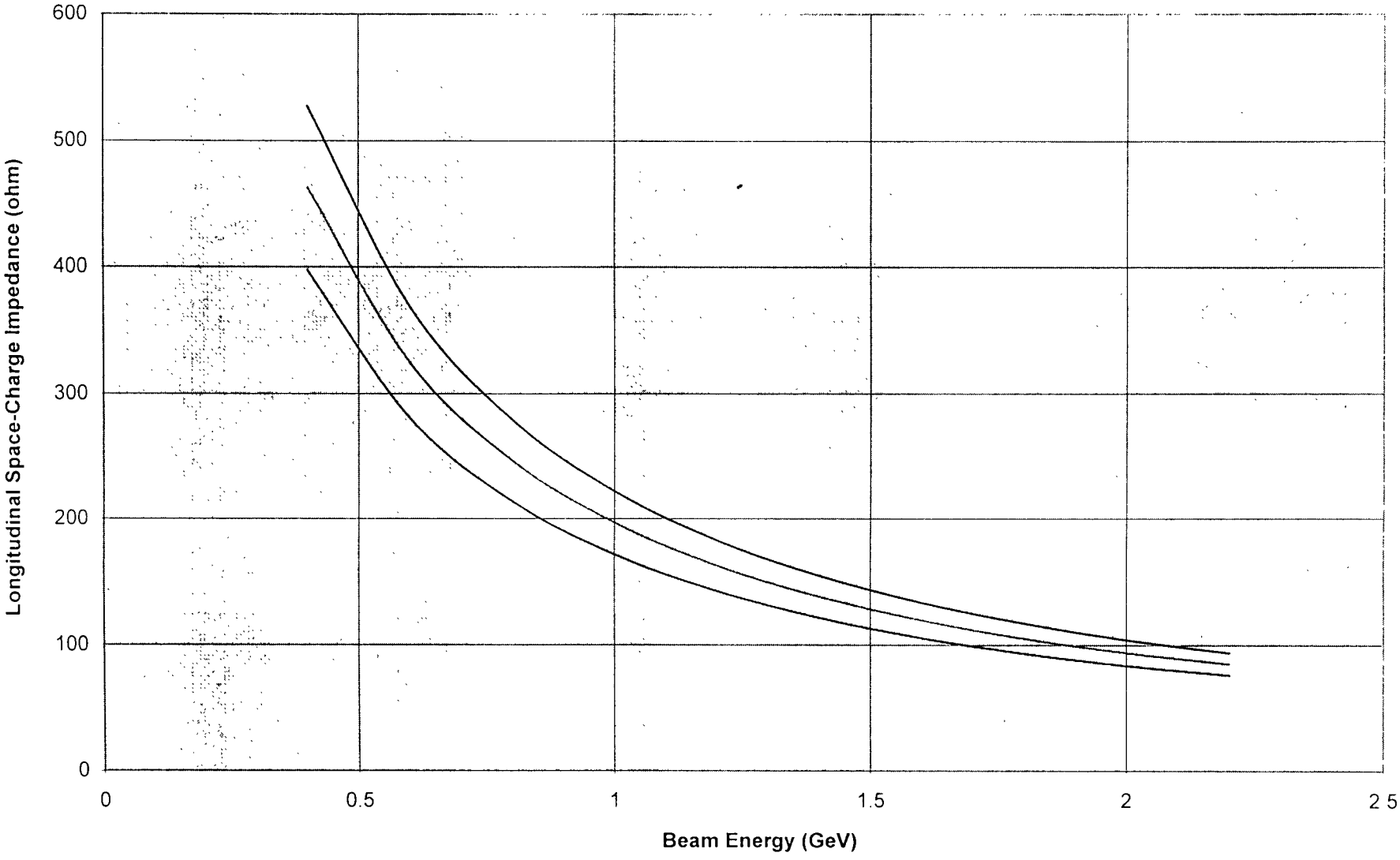
Outline

- **Introduction**
- **Longitudinal Space Charge Impedance:**
  - **Definition and Formula**
  - **Conditions and Limitations**
  - **Corrections**
- **Space Charge Potential**
- **Measurement**
- **Summary**


600
500
400
300
200
100
0
Longitudinal Space-Charge Impedance (ohm)
0
0.5
1
1.5
2
2 5
Beam Energy (GeV)

Definition and Formula

- **Longitudinal space charge complex wave impedance per unit length:**

$$Z_s^*(k,\omega) = -\frac{E_s(k,\omega)}{i_1(k,\omega)} \quad (\Omega/\mathrm{m})$$

For a beam with a uniform circular transverse profile in a round, straight channel under the long wavelength limit:

$$Z_s^*(k,\omega) = i\frac{g}{4\pi}\left(-\frac{ck^2}{\omega} + \frac{\omega}{c}\right)Z_0$$

where g=2*ln(b/a)+α: a geometry factor.

For a linear wave with its phase velocity close to the particle velocity:

$$Z_s^*(\omega) \cong -i\frac{g\omega}{4\pi\beta^2\gamma^2 c}Z_0$$

Definition and Formula (cont.)

- **Longitudinal space charge impedance per perturbation wavelength:**

$$Z_s = Z_s^* \lambda = -i \frac{g}{2\beta\gamma^2} Z_0 \qquad (\Omega)$$

- **Longitudinal space charge impedance in circular accelerators** ($\lambda = 2\pi R/n$):

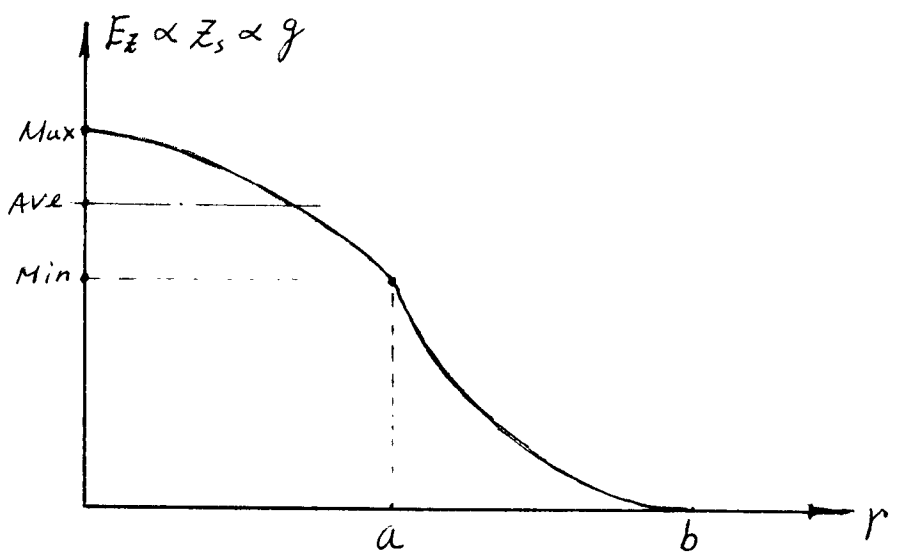
$$\begin{cases} \dfrac{Z_{//}}{n} = -i \dfrac{g}{2\beta\gamma^2} Z_0 \\ g = 2\ln\left(\dfrac{b}{a}\right) + \alpha \end{cases}$$

Geometry Factor g

- **Original formula:**

$$\begin{cases} g = 2\ln(b/a) + \alpha \\ \alpha = 1 - r^2/a^2 \end{cases}$$

- **Physical picture:**



- **Numerical values:**

$$g_{\max} = 2\ln(b/a) + 1$$
$$g_{ave} = 2\ln(b/a) + 1/2$$

Conditions and Limitations

- **Frequencies:**
  - Long wavelength approximation;
- **Beam pipe:**
  - Smooth, conducting wall;
  - Circular cross section;
  - Straight channel;
- **Beam requirements:**
  - Circular cross section, uniform radius;
  - Uniform particle density distribution;
  - Coasting beams;
- **Focusing conditions:**
  - Infinite magnetic fields;

Corrections

- **Non-uniform distributions:**

  - Equivalent beams -

  For a Gaussian density profile:

$$g = 2\ln(b / 2\tilde{x}) + \alpha$$

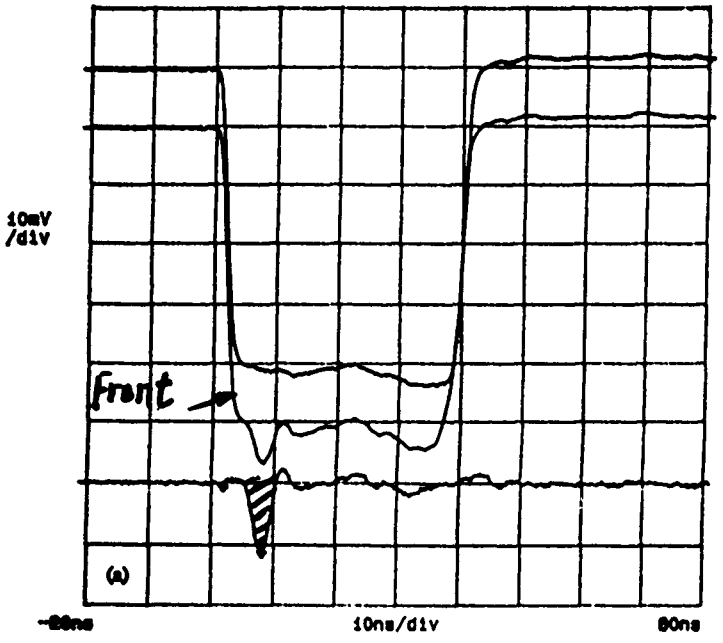
- **Bunched beams:**
  - g(z) dependence:
  - End reflection:

- **Finite magnetic field effect:**

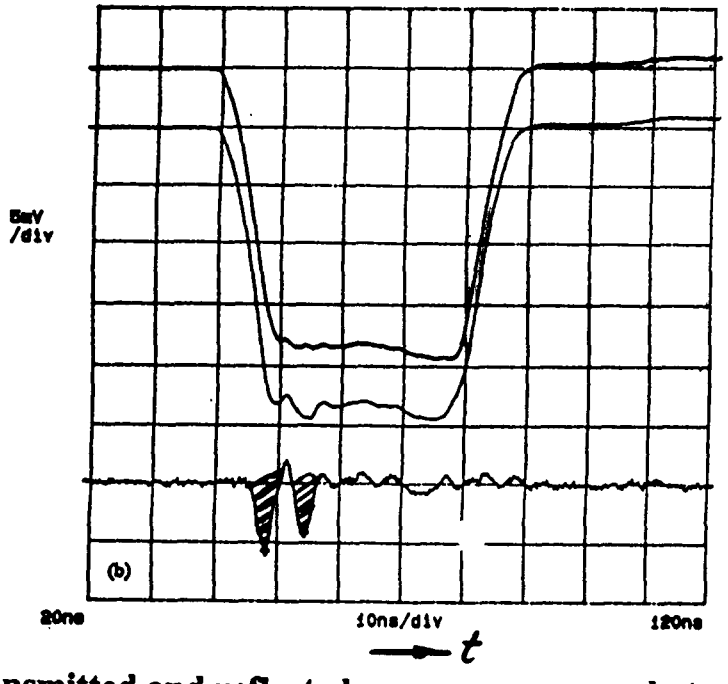
  If the volume density remains constant under a perturbation, it can be shown that

$$g = 2\ln(b / a)$$

Evolution of a fast wave near the beam front end

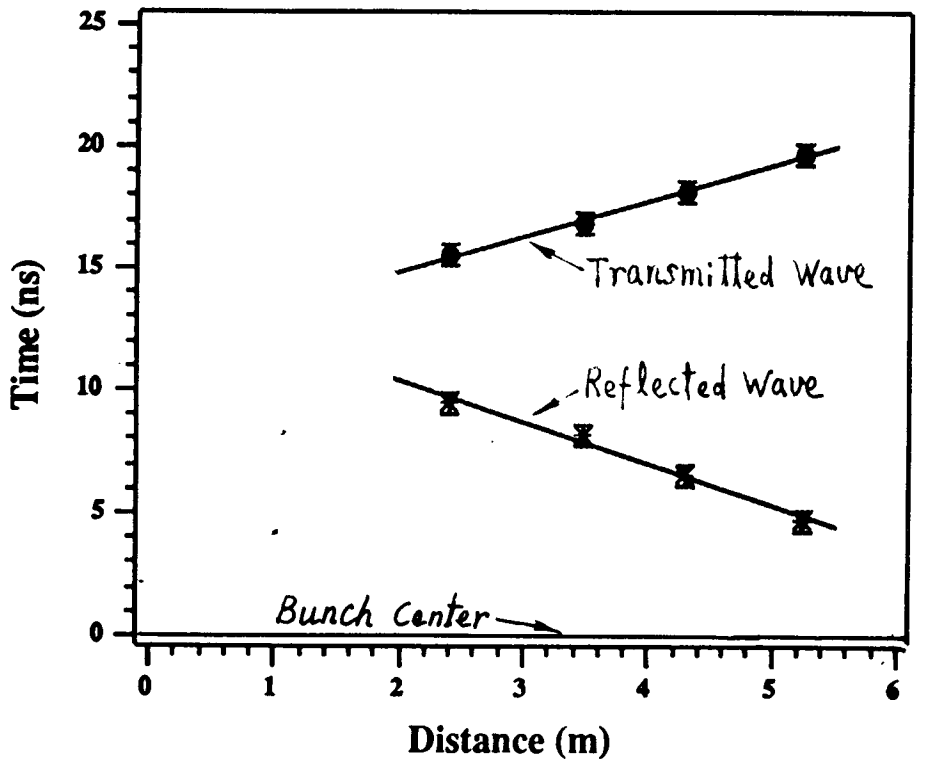


(a) A single fast wave before reaching the beam front end, measured at s=0.624 m;



(b) Transmitted and reflected waves as measured at s=2.39 m;

Propagation speed of reflected and transmitted waves



Time interval between transmitted wave and beam center (dots), and between reflected wave and beam center (stars).

Two different models:

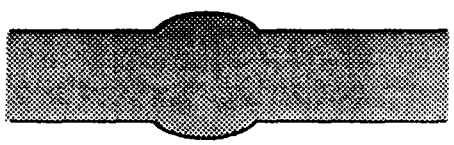
a). Assume constant beam radius and perturbed volume charge density:

$$\alpha = 1 - \left(\frac{r}{a}\right)^2$$

$a = \text{const.}$, $n \sim e^{i(\omega t - kz)}$

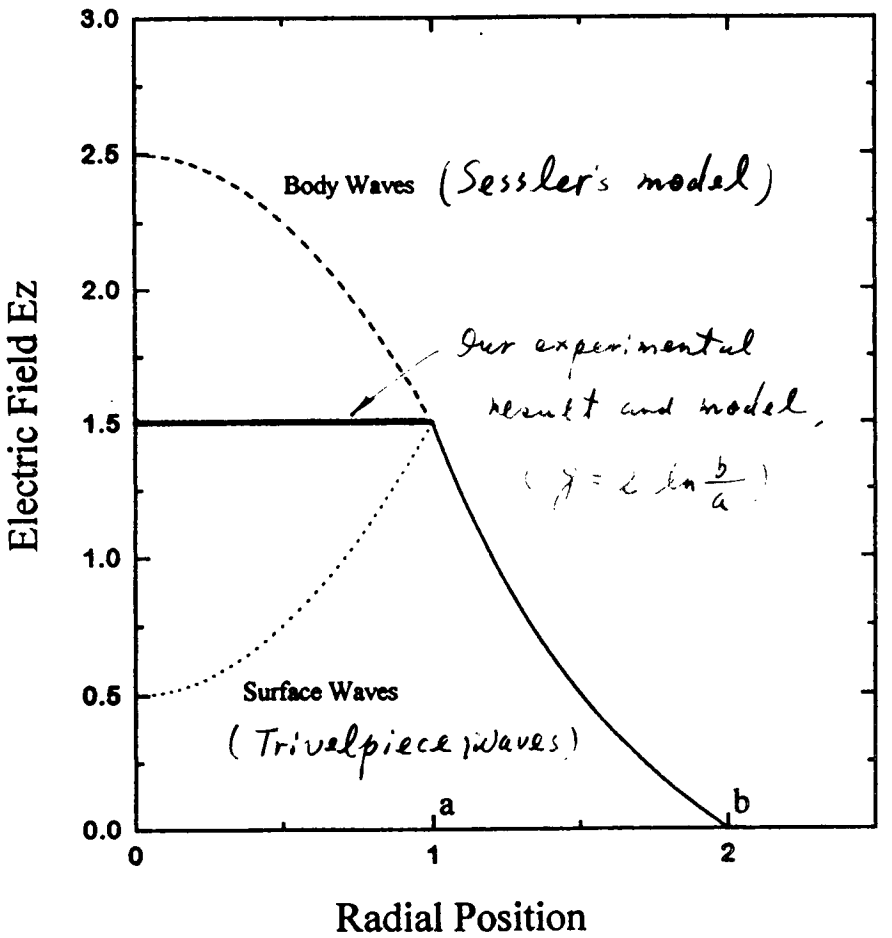
(r = 0 → a: radial position)

b). Assume constant volume density and perturbed beam radius:



$$\alpha = 0$$

$n = \text{const.}$ $a \sim e^{i(\omega t - kz)}$



Perturbed Electrical Field for Different Waves

J. G. Wang and M. Reiser, ~~to appear~~ in Physics of Plasmas, 199
Vol. 5, No. 5, p. 2064

Calculation of Space Charge Potential

- **Similarity to longitudinal space charge impedance**
- **Uniform beams**
- **Other particle distributions**
- **Bunched beams**

Perturbed Electrical Field E_s and Space Charge Potential in a Uniform Beam

- **Space Charge Potential:**

$$\phi(r) = -\frac{g}{4\pi\varepsilon_o}\Lambda$$

$$g = \begin{cases} 2\ln\left(\frac{b}{a}\right) + 1 - \frac{r^2}{a^2} & 0 \le r \le a \\ 2\ln\left(\frac{b}{r}\right) & a \le r \le b \end{cases}$$

- **Perturbed Longitudinal E-field:**

$$E_s(r) = -\frac{g}{4\pi\varepsilon_o}\frac{\partial\Lambda}{\partial z}$$

$$g = \begin{cases} 2\ln\left(\frac{b}{a}\right) + 1 - \frac{r^2}{a^2} & 0 \le r \le a \\ 2\ln\left(\frac{b}{r}\right) & a \le r \le b \end{cases}$$

Geometry Factor g for Bunched Beams

- **Definition:**

$$E_z = -\frac{g}{4\pi\varepsilon_o\gamma^2}\frac{\partial\Lambda(z)}{\partial z}$$

- **Ellipsoidal bunches** with uniform charge density in a conducting cylindrical tube:

$$\Lambda(z) = \Lambda_0\left(1 - \frac{z^2}{z_m^2}\right)$$

The image fields reduce the axial defocusing space-charge force, and increase the radial defocusing space-charge force.

$$\bar{g} \approx 2\ln\left(\frac{b}{a}\right) + \frac{1}{2}$$

$$g_{\max}(0) = 2 \ln \frac{b}{a}, \tag{5.365a}$$

$$g_{\max} \approx 0.67 + 2 \ln \frac{b}{a}. \tag{5.365b}$$

Table 5.3 Geometry parameters $g(0)$, and g for different values of z_m/a and b/a

	$b/a = 1.5$		$b/a = 2$		$b/a = 3$		$b/a = 5$		Free Space
Eccentricity	$g(0)$	g	$g(0)$	g	$g(0)$	g	$g(0)$	g	g_0
1	0.58	0.59	0.63	0.63	0.66	0.66	0.66	0.66	0.67
1.5	0.80	0.85	0.93	0.94	1.01	1.01	1.04	1.04	1.05
2	0.91	1.02	1.14	1.18	1.31	1.31	1.37	1.37	1.39
3	0.94	1.21	1.35	1.48	1.73	1.76	1.90	1.90	1.96
4	0.89	1.30	1.41	1.65	1.98	2.05	2.29	2.30	2.41
5	0.85	1.38	1.40	1.74	2.12	2.24	2.57	2.60	2.79
7.5	0.81	1.38	1.39	1.86	2.19	2.52	2.96	3.08	3.52
10	0.81	1.40	1.39	1.93	2.19	2.63	3.11	3.34	4.06
15	0.81	1.40	1.39	1.97	2.20	2.72	3.19	3.58	4.84
20	0.81	1.41	1.39	1.97	2.20	2.77	3.21	3.68	5.40

Source: Reference 19.

Other Particle Distributions

- **Gaussion distribution as an equivalent uniform beam:**

  for space charge potential calculations:

$$g = 2\ln\left(\frac{b}{2\tilde{x}}\right) + \frac{1}{2}$$

- **"Binomial family of distributions:**

  g-factor for longitudinal space charge force (R. Baartman)

$$\rho = (\mu + 1)\frac{\Lambda}{\pi a^2}\left(1 - \frac{r^2}{a^2}\right)^{\mu}$$

$$\bar{g} = 2\ln\left(\frac{b}{2\tilde{x}}\right) + \frac{1}{2}$$

Measurements

- **Longitudinal Space Charge Impedance of a Uniform Beam:**

  Measure the space charge wave speed

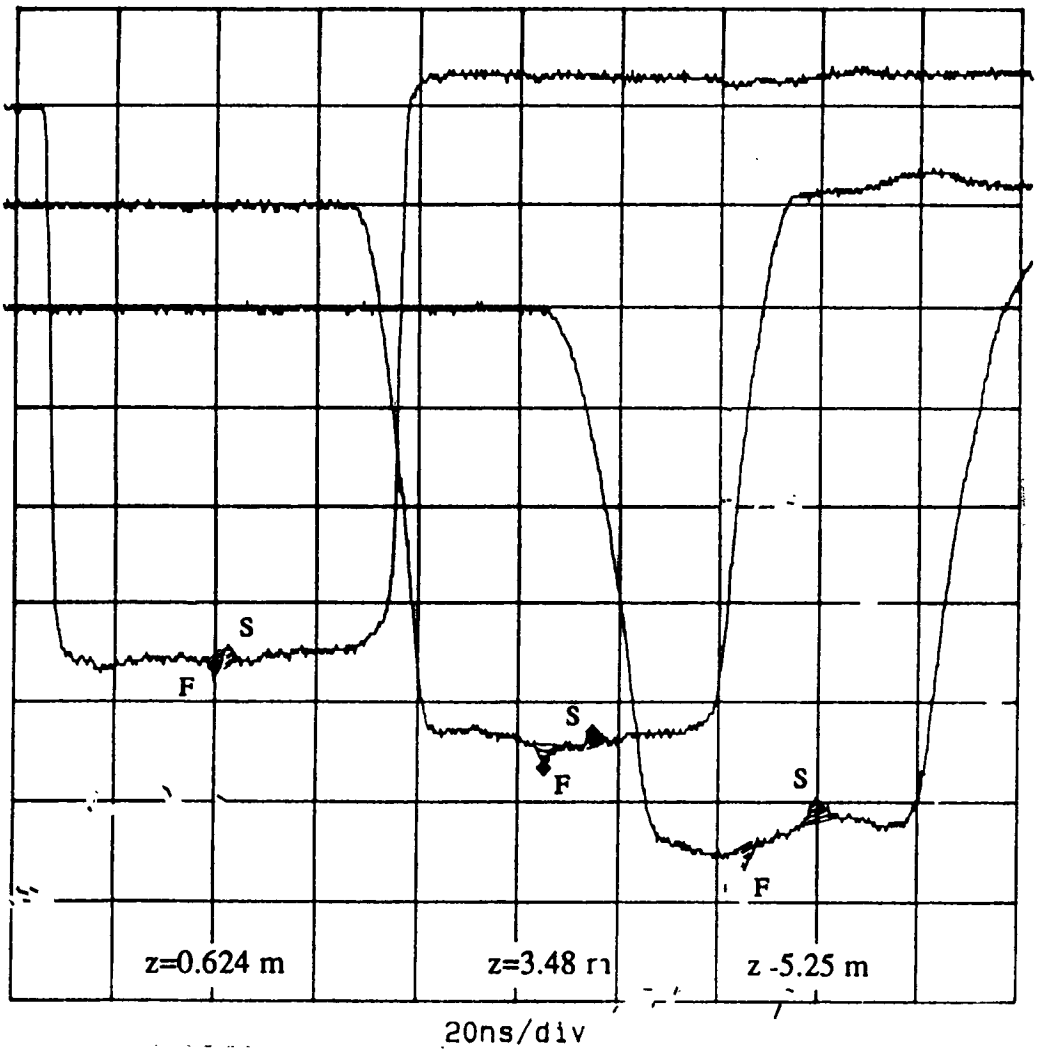
$$c_s = \sqrt{\frac{q\Lambda}{4\pi\varepsilon_0 m} g}$$

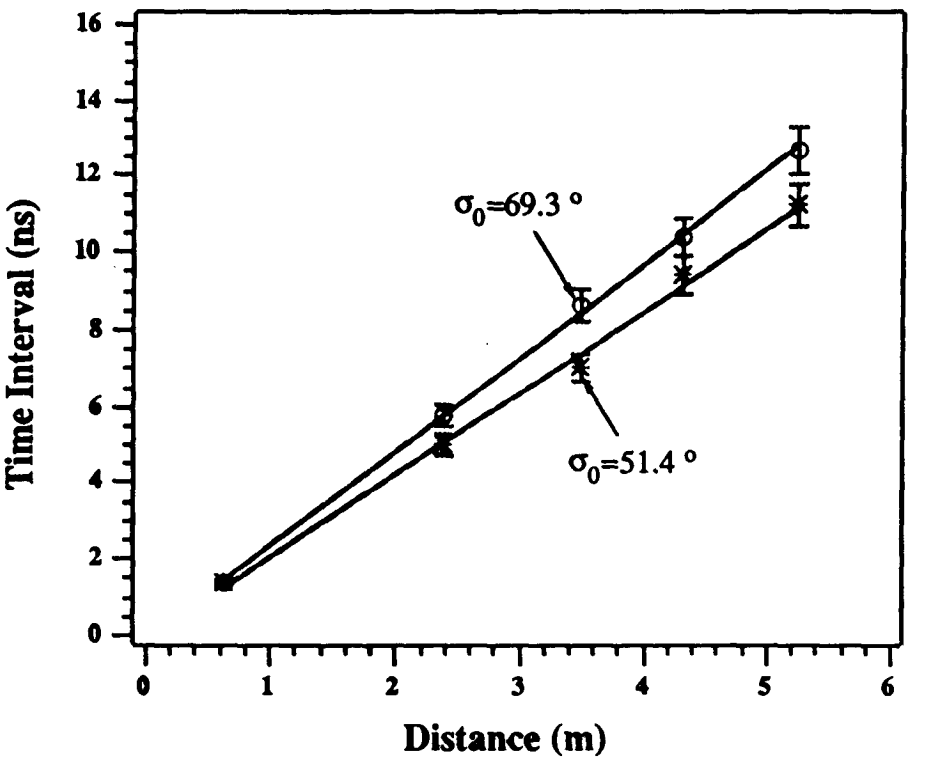
- **Geometry Factor of a Parabolic Bunch:**

  Measure the free expansion of the bunch profile

- **Experimental Measurement:**

Measurement of the g-factor by localized space-charge wave technique



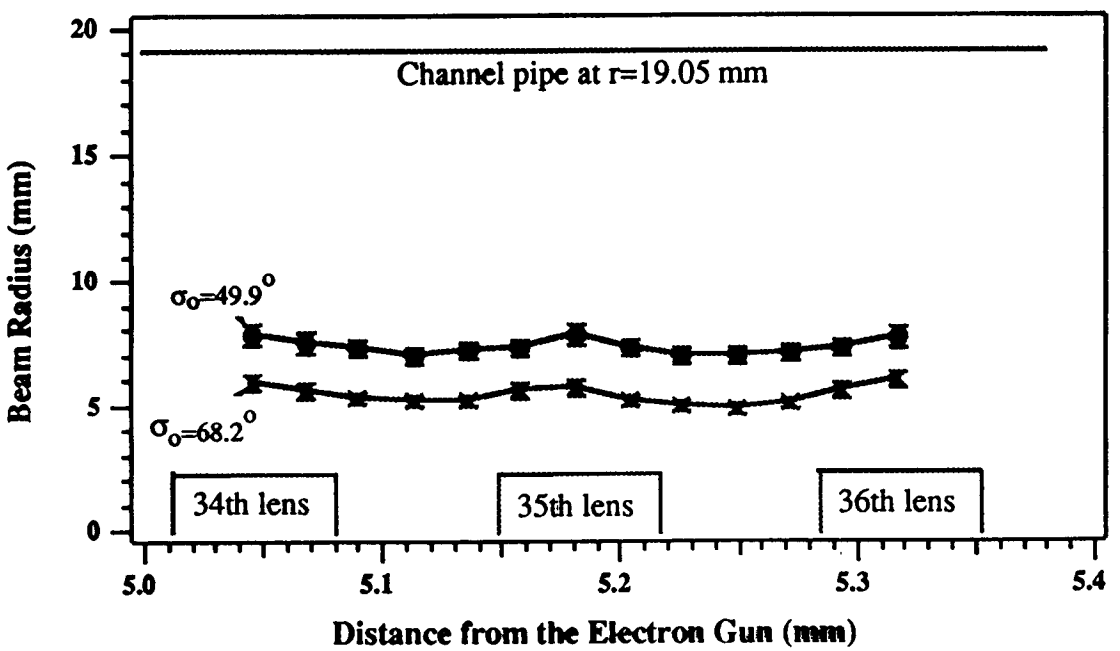


Time interval between two space-charge waves vs. drifting distance for two different phase advances σ_0, as measured by the five current monitors. The solid lines are least-square fits to the experimental data.

The g-factor can be calculated by

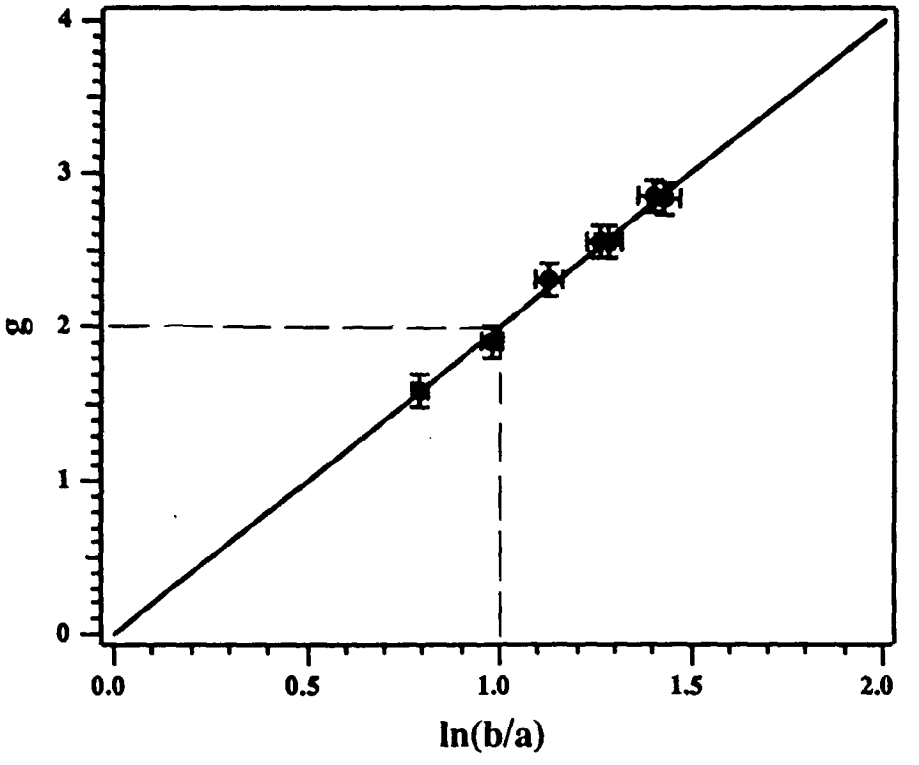
$$\Delta t = \frac{2c_s}{v_0^2 - c_s^2} \cdot z \quad \text{and} \quad c_s = \sqrt{\frac{eg\Lambda_0}{4\pi m \varepsilon_0}}$$

Measurement of the beam radius



Measured beam envelope in the last two periods of the channel for two different phase advances σ_0.

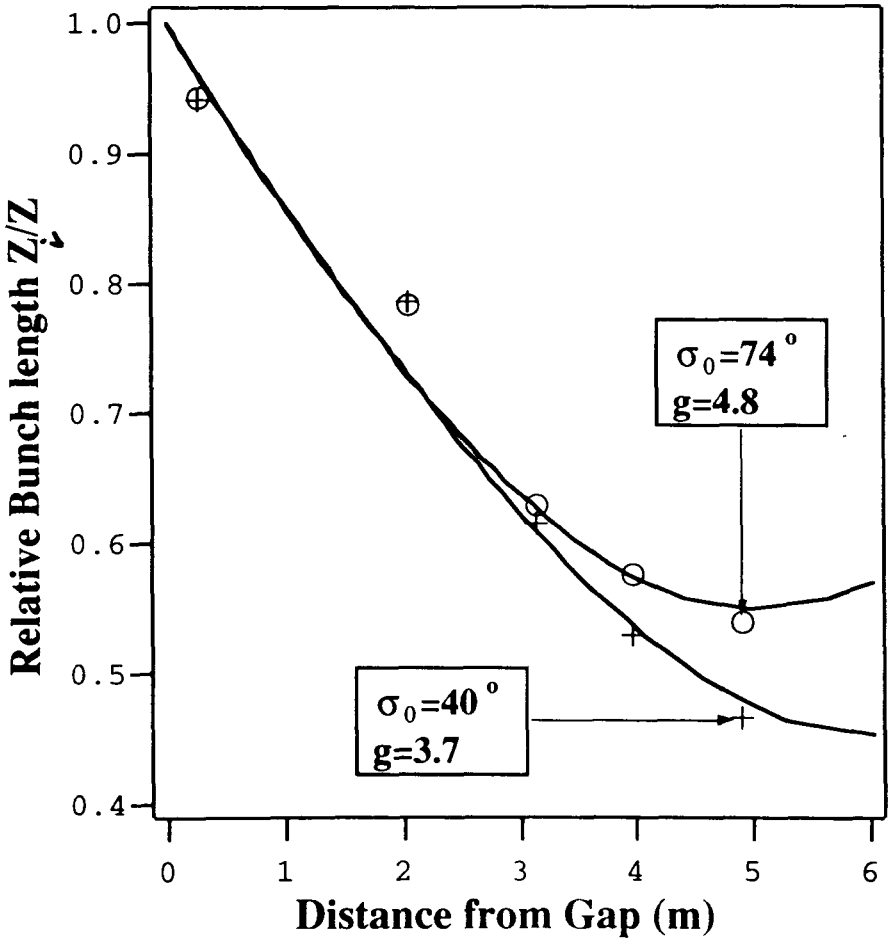
Determination of the g-factor formula



Measured g-factor vs. ln(b/a), where a least-square fitting to the experimental data yields g=2.01 ln(b/a) - 0.01, suggesting that the correct formula for the g-factor should be

$$\underline{g=2 \ln (b/a)} \qquad \text{(i.e. } \alpha=0\text{).}$$

Longitudinal Envelope Evolution with Same Longitudinal Conditions but Different Transverse Focusing Conditions



Summary

- **Review of definition, formula, conditions, and limitations in calculating longitudinal space charge impedance;**

- **Corrections made for non-uniform beams, bunched beams, and finite magnetic focusing effects;**

- **Calculation of space charge potential in various conditions, and comparison with the longitudinal space charge impedance;**

- **Measurements of longitudinal space charge impedance in a uniform beam, and the g-factor for a parabolic bunch.**